University of Sussex

Topics in Geometric and Harmonic Analysis on Symmetric Spaces and Lie Groups

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Declaration

I hereby declare that this thesis has not been and will not be submitted in whole or in part to another University for the award of any other degree.

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Summary

In this thesis, we branch into both harmonic and geometric analysis. Within harmonic analysis, we look at differential actions $\mathscr{L}_{\mathsf{P}}$ on hypergeometric series and Jacobi polynomials, where the latter are known to represent the zonal spherical functions on compact rank-one symmetric spaces. Within geometric analysis, we examine spherical twists as solutions to the Euler-Lagrange system associated with the Dirichlet energy and certain of its nonlinear extensions for sphere-valued mappings in suitable Sobolev spaces.

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Chapter 1

Introduction

This thesis which sits at the intersection of harmonic analysis, geometric analysis and PDEs consists of two main trends. Firstly, a spectral analysis of certain invariant operators, specifically, those built from the Laplace-Beltrami operator, on compact rank-one symmetric spaces of Lie groups. Secondly, a variational analysis of the Dirichlet energy and the resulting harmonic mappings from certain rotationally symmetric domains in the Euclidean space into round spheres with particular emphasis on a geometrically motivated class of mappings, that is, spherical twists, as possible solutions to the resulting system. The first trend relating to symmetric spaces is covered in Chapters 2 and 3 where in addition to harmonic analysis on these spaces interesting questions on the family of Jacobi polynomials and hypergeometric series are addressed and a spectral identity pertaining to the action of a differential operator on the family of Jacobi polynomials is analysed. Here, a new scale of polynomials and a related sequence of scalars describing the aforementioned spectral identity are introduced and various underlying analytic and spectral features of the action including the operator trace and determinant along with their infinite limits and regularisations along with explicit relations and links to hypergeometric parameters are obtained. We recall that the compact rank-one symmetric spaces for the purpose of this thesis entail the n-dimensional sphere $\mathbb{S}^n$, the n-dimensional real projective space $\mathbb{R}\mathbf{P}^n$, the $2n$-dimensional complex projective space $\mathbb{C}\mathbf{P}^n$, the $4n$-dimensional quaternionic space $\mathbb{H}\mathbf{P}^n$ and the 16-dimensional Cayley plane $\mathbf{P}^2(\mathrm{Cay})$ (see Table 2.2). For further details on these spaces and their symmetric structure, see Chapter 2 Section 1 and 2. For the definition and basic properties of Jacobi and hypergeometric series, see Section 2.6.

Indeed, the starting point of the second chapter will be the formulation and proof of the differential identity

$$\mathscr{L}_{\mathsf{P}} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\Big|_{\theta=0} = p_0 + \sum_{m=1}^{\lfloor d/2 \rfloor} p_{2m} \mathscr{R}_m(\lambda_k^{\alpha,\beta})$$

where $\mathscr{L} = \mathsf{P}(d/d\theta)$ is the differential operator associated with the degree $d \geq 2$ polynomial $\mathsf{P} = p_0 + \sum p_i X^i$ whilst $\mathscr{R}_m(X)$ are polynomials relating to the Jacobi polynomials and the hypergeometric series, and $\lambda_k^{\alpha,\beta} = k(k+\alpha+\beta+1)$ are the eigenvalues of the Jacobi operator. The chapter then proceeds by giving explicit descriptions of the polynomials on the right-hand side above and exploiting various implications of this identity to the space of linear differential operators $\mathscr{L}_{\mathsf{P}}$ characterised by the polynomials P.

The second trend is the study of a highly nonlinear PDE system, namely, the harmonic-map equation arising as the Euler-Lagrange equation associated with the Dirichlet energy for suitable sphere-valued Sobolev mappings and examining spherical twists as possible solutions to this system. We also combine this study which in itself is a constrained problem with an associated unconstrained but parameter dependent problem (the so-called penalisation or penalty method) and by bringing in the notion of gamma-convergence, study the relations between these two variational problems in terms of their energy landscapes, structure of minimisers and stationary points, in particular, those in the form of twist mappings.

In the Appendix, computations of the higher order polynomials $\mathscr{R}_m = \mathscr{R}_m(X)$ associated with the Gegenbauer polynomials, Jacobi polynomials and the hypergeometric series will be presented up to order $m = 10$. This will be done by a mixture of substitution and algorithmic derivation. The hypergeometric series will lead to the identification of new coefficients $\mathsf{c}_j^m(a,b,c)$ using the recursive coefficients $\mathsf{d}_{l,j}$, $l, j \geq 1$ and the recursive Bell polynomials b_j^m for $m, j > 0$. The hypergeometric series, with explicit dependence on the parameters a, b and c will give way to the Jacobi polynomials by substituting $a = -k$, $b = k+\alpha+\beta+1$ and $c = \alpha+1$ and the Gegenbauer polynomials by substituting $\alpha = \nu-1/2$ and $\beta = \nu - 1/2$ into the Jacobi polynomials.

Chapter 2

On Determinants and Traces of Matrix Hypergeometric Coefficients and a Differential Action $\mathscr{L}_{\mathsf{P}}$ Pertaining to Rank-One Symmetric Spaces

A differential-spectral identity on the Gauss hypergeometric function is established and an infinite scale of multi-parameter polynomials and matrices describing and encoding the action is introduced. The underlying analytic and spectral features of the action are examined and the operator trace and determinant along with their infinite limits and regularisations as well as explicit relations linking these to the hypergeometric parameters are obtained. Applications to compact symmetric spaces of rank-one and their Laplacian and other related operators, upon invoking the structure of their spherical functions and spectral projections are presented. Other implications including a characterisation of zero action operators as well as extensions to the generalised hypergeometric functions are discussed.

2.1 The Schwartz Kernel of $F(\Delta_{\mathscr{M}})$ on a Compact Rank-One Symmetric Space $\mathscr{M} = G/H$

Let $\mathscr{M}$ be a compact rank-one symmetric space of dimension N and let $-\Delta_{\mathscr{M}}$ denote the positive Laplacian on $\mathscr{M}$. Given $F = F(X)$ for a Hilbert space X in the Borel functional calculus of $-\Delta_{\mathscr{M}}$ the distributional or Schwartz kernel of $F(-\Delta_{\mathscr{M}})$ is given by

$$K_F(\theta) = \sum_{k=0}^{\infty} \frac{M_k(\mathscr{M})}{\mathrm{Vol}(\mathscr{M})} F(\lambda_k) \mathfrak{S}_k(\theta; \mathscr{M}). \tag{2.1}$$

Here $\mathfrak{S}_k = \mathfrak{S}_k(\theta; \mathscr{M})$ are the spherical functions on $\mathscr{M}$, $\lambda_k = \lambda_k(\mathscr{M})$ represent the numerically distinct eigenvalues of $-\Delta_{\mathscr{M}}$ on $\mathscr{M}$, $M_k = M_k(\mathscr{M})$ is the (finite) dimension of the eigenspace associated with λ_k, $\theta = \theta(x, y)$ denotes the distance between the points $x, y \in \mathscr{M}$, and $\mathrm{Vol}(\mathscr{M})$ is the volume of $\mathscr{M}$.

Some specific families of symmetric spaces of interest here include the sphere $\mathbb{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$ and its usual quotient the real projective space $\mathbb{R}\mathbf{P}^n = \mathbb{S}^n/\{\pm\}$; the complex projective space $\mathbb{C}\mathbf{P}^n = \mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(n) \times \mathbf{U}(1))$; the quaternionic projective space $\mathbb{H}\mathbf{P}^n = \mathbf{Sp}(n+1)/\mathbf{Sp}(n) \times \mathbf{Sp}(1)$ and the Cayley plane $\mathbf{P}^2(\mathrm{Cay}) = \mathbf{F}_4/\mathbf{Spin}(9)$.

With regards to the spectral sum (2.1) the relevant geometric and spectral quantities associated with these symmetric spaces $\mathscr{M}$ include, firstly, the radial Laplacian being $\partial^2 + [(\mathsf{a} \cot\theta + \mathsf{b}/2 \cot(\theta/2)]\partial$, with $\partial = \partial_\theta, \partial^2 = \partial_\theta^2$ and the parameters a, b as given in Table 2.1 (*see* below). Secondly, for $k \geq 0$ we have the multiplicity function

$$M_k^n(\mathscr{M}) = \frac{2(k+\varrho)\Gamma(k+2\varrho)\Gamma((\mathsf{a}+1)/2)\Gamma(k+N/2)}{k!\Gamma(2\varrho+1)\Gamma(N/2)\Gamma(k+(\mathsf{a}+1)/2)}, \tag{2.2}$$

of the eigenvalue $\lambda_k^n(\mathscr{M}) = \lambda_k^{(\alpha,\beta)} = (\varrho+k)^2 - \varrho^2$ of $-\Delta_{\mathscr{M}}$ where $\varrho = (\mathsf{a}+\mathsf{b}/2)/2$ and $N = \mathsf{a}+\mathsf{b}+1$ in the simply-connected case. Thirdly, and again in the simply-connected case, we have the volume

$$\mathrm{Vol}(\mathscr{M}) = 2^N \pi^{\frac{N}{2}} \frac{\Gamma((\mathsf{a}+1)/2)}{\Gamma((N+\mathsf{a}+1)/2)}. \tag{2.3}$$

Finally, and most importantly, the spherical functions $\mathfrak{S}_k$ for the symmetric spaces considered here can be described for suitable α, β (*cf.* Table 2.3 below) by the well-known Jacobi polynomials [*see* (2.76)]. As a matter of fact, in the simply-connected case, the spectral sum (2.1) can be rewritten as

$$\begin{aligned} K_F(\theta) &= \sum_{k=0}^{\infty} \frac{M_k^n(\mathscr{M})}{\mathrm{Vol}(\mathscr{M})} F(\lambda_k^n) \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta) \\ &= \frac{(k+\varrho)\Gamma(k+2\varrho)\Gamma(k+N/2)F(\lambda_k^n)}{2^{N-1}\pi^{N/2}k!\Gamma(N/2)\Gamma(k+(\mathsf{a}+1)/2)} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta), \end{aligned} \tag{2.4}$$

Table 2.1: Parameter values and the radial Laplacian for the symmetric space $\mathscr{M}$

$\mathscr{M}$	a	b	ϱ	$\partial^2 + [(\mathsf{a}\cot\theta + \mathsf{b}/2\cot(\theta/2)]\partial$
$\mathbb{S}^n$	$n-1$	0	$(n-1)/2$	$\partial^2 + [(n-1)\cot\theta]\partial$
$\mathbb{R}\mathbf{P}^n$	$n-1$	0	$(n-1)/2$	$\partial^2 + [(n-1)\cot\theta]\partial$
$\mathbb{C}\mathbf{P}^n$	1	$2(n-1)$	$n/2$	$\partial^2 + [\cot\theta + (n-1)\cot(\theta/2)]\partial$
$\mathbb{H}\mathbf{P}^n$	3	$4(n-1)$	$(2n+1)/2$	$\partial^2 + [3\cot\theta + 2(n-1)\cot(\theta/2)]\partial$
$\mathbf{P}^2(\mathrm{Cay})$	7	8	$11/2$	$\partial^2 + [7\cot\theta + 4\cot(\theta/2)]\partial$

(with the modification $\mathfrak{S}_k = \mathscr{P}_{2k}^{((n-2)/2,(n-2)/2)}(\cos\theta) = \mathscr{C}_{2k}^{(n-1)/2}(\cos\theta)$ and $M_k^n(\mathscr{M})$, $\mathrm{Vol}(\mathscr{M})$ from Table 2.1 in the non simply-connected case $\mathscr{M} = \mathbb{R}\mathbf{P}^n$). Next for the sake of various applications, localising θ to a neighbourhood of the origin, it is seen that subject to sufficient regularity, the Maclaurin coefficients of the Schwartz kernel K_F in (2.4), and subsequently the formal Maclaurin expansion of K_F about $\theta = 0$ take the form

$$b_{2j}[F] = \mathrm{Vol}(\mathscr{M}) \frac{\partial^{2j}}{\partial\theta^{2j}} K_F \Big|_{\theta=0}, \qquad \sum_{j=0}^{\infty} \frac{\theta^{2j}}{(2j)!} \frac{\partial^{2j}}{\partial\theta^{2j}} K_F \Big|_{\theta=0} = \sum_{j=0}^{\infty} \frac{\theta^{2j}}{(2j)!} \frac{b_{2j}[F]}{\mathrm{Vol}(\mathscr{M})}. \tag{2.5}$$

Now let $\mathsf{P}_d(X) = p_0 + \sum_{i=1}^{d} p_i X^i$ be a polynomial of degree $d \geq 2$ and consider the associated differential operator $\mathscr{L}_\mathsf{P} = \mathsf{P}_d(d/d\theta) = p_0 + \sum_{i=1}^{d} p_i d^i/d\theta^i$. Then by specialising Theorem 2.2.1 below to the case of Jacobi polynomials we have

$$\begin{aligned} \mathscr{L}_\mathsf{P} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\Big|_{\theta=0} &= p_0 + \sum_{m=1}^{\lfloor d/2 \rfloor} p_{2m} \sum_{j=1}^{m} \mathsf{c}_j^m [\lambda_k^{\alpha,\beta}]^j \\ &= p_0 + \sum_{m=1}^{\lfloor d/2 \rfloor} p_{2m} \mathscr{R}_m(\lambda_k^{\alpha,\beta}). \end{aligned} \tag{2.6}$$

Here $\lambda_k^{\alpha,\beta} = k(\alpha+\beta+k+1)$ are the eigenvalues of the Jacobi operator [*cf.* (2.73)], $\mathsf{c}_j^m = \mathsf{c}_j^m(\alpha,\beta)$ are explicitly computable scalars and $\mathscr{R}_m(X)$ is the degree m polynomial $\mathscr{R}_m(X) = \mathscr{R}_m(X;\alpha,\beta) = \sum_{j=1}^{m} \mathsf{c}_j^m(\alpha,\beta) X^j$.

By specialising to the case $\mathscr{L}_\mathsf{P} = d^{2j}/d\theta^{2j}$ and referring to (2.4)-(2.5) above it follows at once that the Maclaurin spectral coefficients associated with the Schwartz kernel K_F, are in turn given by $b_0[F] = \mathrm{tr}[F(-\Delta_\mathscr{M})]$ and for $j \geq 1$ by

$$b_{2j}[F] = \mathrm{Vol}(\mathscr{M}) \frac{\partial^{2j}}{\partial\theta^{2j}} K_F(\theta)\Big|_{\theta=0} = \mathrm{tr}[\mathscr{R}_j F(-\Delta_\mathscr{M})], \tag{2.7}$$

where tr stands for operator trace. This gives one useful application of the action identity (2.6) and the polynomials $\mathscr{R}_m(X)$ and their coefficients c_j^m (with $1 \le j \le m$). In the remainder of this chapter we extend the action identity (2.6) to the context of the hypergeometric function $F(a, b; c; z)$, where a huge family of special functions and orthogonal polynomials of interest in analysis and mathematical physics, including the scale of Jacobi, Gegenbauer and Legendre polynomials are covered as particular examples (*see* the Appendix and Section 2.5). We then move on to studying the action identity (2.6)-(2.13) more formally on a naturally associated space of polynomials and obtain concrete spectral results on the corresponding matrix operators. These in particular entail a description of the spectrum, trace and determinants of the resulting matrices Q_l in Section 2.3 and their infinite limits. We also characterise the space of zero action operators $\mathscr{L}_{\mathsf{P}}$ by looking at the kernel of a corresponding functional Λ_d in Section 2.4. Applications and implications of these results along with various explicit computations are then discussed in the context of compact rank-one symmetric spaces $\mathscr{M}$ described above.

Table 2.2: Spectral data for rank-one symmetric spaces with dimension N

$\mathscr{M}$	N	λ_k^n	$M_k^n(\mathscr{M})$	$\mathrm{Vol}(\mathscr{M})$
$\mathbb{S}^n$	n	$k(k+n-1)$	$\frac{(2k+n-1)(k+n-2)!}{k!(n-1)!}$	$\frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}$
$\mathbb{R}\mathbf{P}^n$	n	$2k(2k+n-1)$	$\frac{(4k+n-1)(2k+n-2)!}{(2k)!(n-1)!}$	$\frac{\pi^{(n+1)/2}}{\Gamma((n+1)/2)}$
$\mathbb{C}\mathbf{P}^n$	$2n$	$k(k+n)$	$\frac{2k+n}{n}\left(\frac{\Gamma(k+n)}{\Gamma(n)k!}\right)^2$	$\frac{4^n\pi^n}{n!}$
$\mathbb{H}\mathbf{P}^n$	$4n$	$k(k+2n+1)$	$\frac{(2k+2n+1)(k+2n)}{(2n)(2n+1)(k+1)}\left(\frac{\Gamma(k+2n)}{k!\Gamma(2n)}\right)^2$	$\frac{4^{2n}\pi^{2n}}{(2n+1)!}$
$\mathbf{P}^2(\mathrm{Cay})$	16	$k(k+11)$	$6(2k+11)\frac{\Gamma(k+8)\Gamma(k+11)}{7!11!k!\Gamma(k+4)}$	$\frac{3!(4\pi)^8}{11!}$

2.2 A Differential Action and the Hypergeometric Coefficients $(\mathsf{c}_j^m : 1 \le j \le m)$

In this section we formulate and prove a differential operator action identity on the hypergeometric function that the future development of the chapter heavily relies upon. (For a quick account on the main properties of the hypergeometric function used here the reader is invited to consult the Appendix at the end of the paper.) Before proceeding on to Theorem 2.2.1 we pause briefly to introduce some notation and terminology for later use. First for a given finite sequence of scalars $s_0, s_1, \dots, s_{j-1}$ (with $j \geq 1$) we define the associated numbers $\mathsf{d}_{0,j}, \mathsf{d}_{1,j}, \dots, \mathsf{d}_{j,j}$ as the coefficients of the polynomial $s(X) = (X+s_0)(X+s_1)\dots(X+s_{j-1}) = \mathsf{d}_{0,j} + \mathsf{d}_{1,j}X + \dots \mathsf{d}_{j,j}X^j$, that is,

$$s(X; s_0, s_1, \dots, s_{j-1}) = \prod_{p=0}^{j-1} (X + s_p) = \sum_{l=0}^{j} \mathsf{d}_{l,j} X^l. \tag{2.8}$$

Thus here $\mathsf{d}_{l,j} = \mathsf{d}_{l,j}(s_0, s_1, \dots, s_{j-1})$ are the sums of the products of the $\binom{j}{l}$ combinations of $j-l$ values s_p from $(s_0, s_1, \dots, s_{j-1})$, or more technically, the elementary symmetric polynomials in $s_0, s_1 \dots, s_{j-1}$. In particular we have

$$\mathsf{d}_{0,j} = s_0 s_1 \dots s_{j-1}, \qquad \mathsf{d}_{j-1,j} = s_0 + s_1 \dots + s_{j-1}, \qquad \mathsf{d}_{j,j} = 1. \tag{2.9}$$

The next lemma relates the product of Pochhammer symbols [*cf.* (2.68)] to the scalars $\mathsf{d}_{l,j}$ introduced above. For a generalisation of this identity *see* Section 2.5.

Lemma 2.2.1. *For a given pair of scalars a and b and $j \geq 1$ the product $(a)_j(b)_j$ can be written as a polynomial of degree j in $X = ab$ as*

$$(a)_j(b)_j = \prod_{p=0}^{j-1} \left(X + \underbrace{p(a+b+p)}_{s_p} \right) = \sum_{l=1}^{j} \mathsf{d}_{l,j} X^l = s(X; s_0, s_1, \dots, s_{j-1}), \tag{2.10}$$

where the scalars $\mathsf{d}_{l,j} = \mathsf{d}_{l,j}(a+b)$ are defined in (2.8) *by setting $s_p = p(a+b+p)$.*

Proof. Referring to (2.68) and forming the product $(a)_j(b)_j$ we can write

$$\begin{aligned}(a)_j(b)_j &= \left\{ \prod_{k=0}^{j-1} (a+k) \right\} \left\{ \prod_{l=0}^{j-1} (b+l) \right\} \\ &= \prod_{p=0}^{j-1} (a+p)(b+p) = \prod_{p=0}^{j-1} \left(ab + p(a+b+p) \right). \end{aligned} \tag{2.11}$$

The desired conclusion now follows by noting (2.8). Note that since $s_0 = 0$ we have $\mathsf{d}_{0,j} = 0$ and so the sum in (2.8) here starts from $l = 1$. $\square$

The next theorem describes a specific action of the differential operator $\mathscr{L}_{\mathsf{P}}$ on the hypergeometric function $F(a, b; c; z)$ by connecting to a natural class of polynomials $\mathscr{R}_m = \mathscr{R}_m(X; a, b, c)$ $(m \geq 1)$. These polynomials are explicitly described and their coefficients $\mathsf{c}_j^m = \mathsf{c}_j^m(a, b, c)$ (with $1 \leq j \leq m$) characterised in a fully computable way. Due to the significance of these coefficients throughout the paper they are hereafter referred to as the *hypergeometric coefficients*.

Theorem 2.2.1. *Let $\mathsf{P}_d(X) = \sum_{0\le i\le d} p_i X^i$ be a polynomial of degree $d \ge 2$ and let $\mathscr{L}_{\mathsf{P}}$ denote the differential operator*

$$\mathscr{L}_{\mathsf{P}} = \mathsf{P}_d(d/d\theta) = p_0 + \sum_{i=1}^{d} p_i d^i/d\theta^i. \tag{2.12}$$

Then for $a, b, c \in \mathbb{C}$ with $c \neq 0, -1, -2, \cdots$ the action of $\mathscr{L}_{\mathsf{P}}$ on the hypergeometric function $F(a, b, c, ; z)$ satisfies the relation

$$(\mathscr{L}_{\mathsf{P}} F)\left(a, b; c; \frac{1-\cos\theta}{2}\right)\bigg|_{\theta=0} = p_0 + \sum_{m=1}^{\lfloor d/2 \rfloor} p_{2m} \mathscr{R}_m(-ab). \tag{2.13}$$

Here $\mathscr{R}_m(X) = \mathscr{R}_m(X; a, b, c)$ is the degree $m \ge 1$ polynomial

$$\mathscr{R}_m(X) = \mathscr{R}_m(X; a, b, c) = \sum_{j=1}^{m} \mathsf{c}_j^m(a, b, c) X^j, \tag{2.14}$$

with coefficients c_j^m, $1 \le j \le m$ (the hypergeometric coefficients) given explicitly by

$$\mathsf{c}_j^m = \mathsf{c}_j^m(a, b, c) = \sum_{i=j}^{m} (-1)^{i+j} \mathsf{b}_i^m \mathsf{d}_{j,i} \left\{ 2^i \prod_{p=0}^{i-1} (c+p) \right\}^{-1}. \tag{2.15}$$

Furthermore in (2.15) the scalars b_j^m are defined recursively by the relation

$$\mathsf{b}_j^m = \begin{cases} (-1)^m, & \text{if } j = 1, \\ -(j^2 \mathsf{b}_j^{m-1} + (2j-1)\mathsf{b}_{j-1}^{m-1}), & \text{if } 2 \le j \le m, \\ 0, & \text{if } j > m, \end{cases} \tag{2.16}$$

and $\mathsf{d}_{j,i} = \mathsf{d}_{j,i}(a+b)$ are the scalars defined in (2.8) with $s_p = p(a+b+p)$.

Proof. We begin by noting that since $F(a, b; c; (1-\cos\theta)/2)$ is an even function of θ, all its derivatives of odd order at zero vanish. Hence, when applying $\mathscr{L}_{\mathsf{P}}$ to $F(a, b; c; (1-\cos\theta)/2)$ and evaluating at $\theta = 0$, we have

$$\begin{aligned} (\mathscr{L}_{\mathsf{P}} F)\left(a, b; c; \frac{1-\cos\theta}{2}\right)\bigg|_{\theta=0} &= p_0 + \sum_{i=1}^{d} p_i \frac{d^i}{d\theta^i} F\left(a, b; c; \frac{1-\cos\theta}{2}\right)\bigg|_{\theta=0} \\ &= p_0 + \sum_{m=1}^{\lfloor d/2 \rfloor} p_{2m} \frac{d^{2m}}{d\theta^{2m}} F\left(a, b; c; \frac{1-\cos\theta}{2}\right)\bigg|_{\theta=0}. \end{aligned} \tag{2.17}$$

We are now in a position to use (2.80) with $f(\cos\theta) = F(a, b; c; (1-\cos\theta)/2)$. This when combined with the differential identity (2.71) gives

$$\begin{aligned} \frac{d^{2m}}{d\theta^{2m}} F\left(a, b; c; \frac{1-\cos\theta}{2}\right)\bigg|_{\theta=0} &= \left\{ \sum_{j=1}^{m} \frac{\mathsf{b}_j^m}{(-2)^j} \frac{d^j}{dz^j} F(a, b; c; z) \right\}\bigg|_{z=0} \\ {\scriptstyle \text{via (2.71)}} &= \sum_{j=1}^{m} \frac{\mathsf{b}_j^m}{(-2)^j} \frac{(a)_j (b)_j}{(c)_j} F(a+j, b+j; c+j, 0) \\ &= \sum_{j=1}^{m} \frac{\mathsf{b}_j^m}{(-2)^j} \frac{(a)_j (b)_j}{(c)_j}. \end{aligned} \tag{2.18}$$

Next making use of Lemma 2.2.1 on the product $(a)_j(b)_j$ and writing $\mathscr{C}^j(c) = (-2)^j(c)_j = (-2)^j \prod_{p=0}^{j-1}(c+p)$, upon substituting into (2.18) we have,

$$\begin{aligned}\frac{d^{2m}}{d\theta^{2m}} F\left(a, b; c; \frac{1-\cos\theta}{2}\right)\Bigg|_{\theta=0} &= \sum_{j=1}^{m} \frac{\mathsf{b}_j^m}{\mathscr{C}^j(c)}(a)_j(b)_j \\ &= \sum_{j=1}^{m} \frac{\mathsf{b}_j^m}{\mathscr{C}^j(c)} \sum_{l=1}^{j} \mathsf{d}_{l,j}(ab)^l. \end{aligned} \tag{2.19}$$

Expanding the last sum and rearranging in powers of ab we have

$$\begin{aligned}\frac{d^{2m}}{d\theta^{2m}} F\left(a, b; c; \frac{1-\cos\theta}{2}\right)\Bigg|_{\theta=0} &= \sum_{j=1}^{m}(ab)^j \left(\sum_{i=j}^{m} \frac{\mathsf{b}_i^m}{\mathscr{C}^i(c)}\mathsf{d}_{j,i}\right) \\ &= \sum_{j=1}^{m}(-ab)^j \left((-1)^j \sum_{i=j}^{m} \frac{\mathsf{b}_i^m}{\mathscr{C}^i(c)}\mathsf{d}_{j,i}\right) \\ &= \sum_{j=1}^{m}(-ab)^j \mathsf{c}_j^m(a,b,c) = \mathscr{R}_m(-ab), \end{aligned} \tag{2.20}$$

where we have written $(-1)^j \sum_{i=j}^{m} \mathsf{b}_i^m \mathsf{d}_{j,i}[\mathscr{C}^i(c)]^{-1} = \mathsf{c}_j^m(a,b,c)$. The conclusion follows at once upon substituting back into (2.17). $\square$

Notice that in view of $d_{m,m} = 1$ [*cf.* (2.9)-(2.15)], the leading hypergeometric coefficient c_m^m – unlike the sub-leading coefficients c_j^m with $j < m$ – depends only on the parameter c (and not on a, b). Furthermore the dependence of the sub-leading coefficients on the parameters a, b is only through the sum $a + b$ (*see* (2.11) and the description of $\mathsf{d}_{j,i}$ in Theorem 2.2.1). Now, for the sake of future reference, a direct verification reveals that the first few polynomials $\mathscr{R}_m = \mathscr{R}_m(X; a, b, c)$ are given explicitly by

$$\begin{aligned}\mathscr{R}_1(X; a, b, c) &= -\frac{X}{2c}, \ \mathscr{R}_2(X; a, b, c) = \frac{3X^2 + (2c - 1 - 3(a+b))X}{4c(c+1)}, \\ \mathscr{R}_3(X; a, b, c) &= \frac{-15X^3 + (-30c + 15 + 45(a+b))X^2}{8c(c+1)(c+2)} \\ &\quad - \frac{[30(a+b)(a+b+1-c) + 4c^2 - 18c + 8]X}{8c(c+1)(c+2)}. \end{aligned} \tag{2.21}$$

We will make extensive use of these polynomials in the calculations towards the end of Section 2.3 and Section 2.4. Table 2.3 below describes the parameters a, b, c and the corresponding α, β for each of the symmetric spaces listed earlier in Section 2.1. Note that here $\alpha = (N-2)/2$ and $\beta = (\mathsf{a} - 1)/2$.

2.3 The Operator $\mathbb{T}_d$ and the Triangular Matrix Q_l

According to Theorem 2.2.1 the action of $\mathscr{L}_\mathsf{P}$ on the hypergeometric function $F = F(a, b, c; z)$ is a linear combination of the polynomials $\mathscr{R}_m = \mathscr{R}_m(X; a, b, c)$ evaluated at $X = -ab$.

Table 2.3: Parameter values for rank-one symmetric spaces

$\mathscr{M}$	a	b	c	$-ab$	α	β
$\mathbb{S}^n$	$-k$	$k+n-1$	$n/2$	$k(k+n-1)$	$n/2-1$	$n/2-1$
$\mathbb{R}\mathbf{P}^n$	$-2k$	$2k+n-1$	$n/2$	$2k(2k+n-1)$	$n/2-1$	$n/2-1$
$\mathbb{C}\mathbf{P}^n$	$-k$	$k+n$	n	$k(k+n)$	$n-1$	0
$\mathbb{H}\mathbf{P}^n$	$-k$	$k+2n+1$	$2n$	$k(k+2n+1)$	$2n-1$	1
$\mathbf{P}^2(\mathrm{Cay})$	$-k$	$k+11$	8	$k(k+11)$	7	3

This naturally prompts the introduction of an operator family $\mathbb{T}_d : \mathsf{P} \mapsto \mathbb{T}_d(\mathsf{P}; a, b, c)$ (with $d \geq 2$) as given below.

Definition 2.3.1. *For a, b, c as above and $d \geq 2$ we denote by $\mathbb{T} = \mathbb{T}_d$ the operator acting on the space of polynomials P_d as in Theorem 2.2.1 defined by*

$$\mathbb{T}_d : \mathsf{P}_d(X) = p_0 + \sum_{j=1}^{d} p_j X^j \mapsto Q_{\lfloor d/2 \rfloor}(X) = p_0 + \sum_{m=1}^{\lfloor d/2 \rfloor} p_{2m} \mathscr{R}_m(X), \tag{2.22}$$

where $\mathscr{R}_m = \mathscr{R}_m(X; a, b, c)$ are the polynomials defined in Theorem 2.2.1.

Note that with the aid of the above definition the conclusion of Theorem 2.2.1 can be rewritten in the form

$$(\mathscr{L}_{\mathsf{P}} F)\left(a, b; c; \frac{1-\cos\theta}{2}\right)\Big|_{\theta=0} = \mathbb{T}_d[\mathsf{P}; a, b, c](-ab) = Q_{\lfloor d/2 \rfloor}(-ab). \tag{2.23}$$

For the sake of clarity and to fix ideas let us pause briefly to look into this in more detail for certain smaller values of d. Indeed using the formulation of the polynomials $\mathscr{R}_m$ in Theorem 2.2.1 and the explicit description for $1 \leq m \leq 3$ in (2.21) it is seen that for $2 \leq d \leq 3$ we have $Q(X) = \mathbb{T}_d[\mathsf{P}](X) = p_0 + p_2\mathscr{R}_1(X) = p_0 + p_2 \mathsf{c}_1^1 X = p_0 - p_2 X/2c$; in particular $Q(-ab) = p_0 + p_2 ab/2c$. Likewise for $4 \leq d \leq 5$ and again with $Q(X) = \mathbb{T}_d[\mathsf{P}](X)$ we have

$$\begin{aligned} Q(X) &= p_0 + p_2\mathscr{R}_1 + p_4\mathscr{R}_2 = p_0 + p_2\mathsf{c}_1^1 X + p_4(\mathsf{c}_2^2 X^2 + \mathsf{c}_1^2 X) \\ &= p_0 + (p_4\mathsf{c}_1^2 + p_2\mathsf{c}_1^1)X + p_4\mathsf{c}_2^2 X^2 \\ &= p_0 - \frac{[2c(p_2 - p_4) + (3a + 3b + 1)p_4 + 2p_2]X + 3p_4 X^2}{4c(c+1)}, \end{aligned} \tag{2.24}$$

hence $Q(-ab) = p_0 + ab/(2c)p_2 + (3a^2b^2 + 3a^2b + 3ab^2 - 2abc + ab)/[4c(c+1)]p_4$. Returning now to (3.8) and the description of the polynomial $Q = \mathbb{T}_d[\mathsf{P}]$ it is evident that

$$\begin{aligned}\mathbb{T}_d[\mathsf{P}] = Q(X) &= p_0 + \sum_{m=1}^{\lfloor d/2 \rfloor} p_{2m} \mathscr{R}_m(X) \\ &= p_0 + \sum_{m=1}^{\lfloor d/2 \rfloor} \sum_{j=1}^{m} p_{2m} \mathsf{c}_j^m X^j \\ &= p_0 + \sum_{j=1}^{\lfloor d/2 \rfloor} \left[\sum_{m=j}^{\lfloor d/2 \rfloor} p_{2m} \mathsf{c}_j^m \right] X^j. \end{aligned} \tag{2.25}$$

Hence setting $l = \lfloor d/2 \rfloor$ with $l \geq 1$, the latter, upon introducing the $(l+1) \times (l+1)$ lower triangular matrix Q_l can be written as $Q(X) = \langle \mathsf{Q}_l \mathbf{P}, \mathbf{X} \rangle$, that is,

$$Q(X) = \left\langle \begin{bmatrix} \mathsf{c}_l^l & 0 & 0 & \cdots & 0 \\ \mathsf{c}_{l-1}^l & \mathsf{c}_{l-1}^{l-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathsf{c}_1^l & \mathsf{c}_1^{l-1} & \cdots & \mathsf{c}_1^1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{2l} \\ p_{2(l-1)} \\ \vdots \\ p_2 \\ p_0 \end{bmatrix}, \begin{bmatrix} X^l \\ X^{l-1} \\ \vdots \\ X \\ 1 \end{bmatrix} \right\rangle. \tag{2.26}$$

Here $\mathbf{P} = (p_{2l}, \cdots, p_2, p_0)$ is the vector of the *even* coefficients of P_d while $\mathbf{X} = (X^l, \cdots, X, 1)$. Moreover referring to the matrix Q_l it is seen that the eigenvalues are given by the leading hypergeometric coefficients $\mathsf{c}_l^l, \mathsf{c}_{l-1}^{l-1}, \cdots, \mathsf{c}_1^1, 1$.

Theorem 2.3.1. *Let $\mathsf{Q}_l = \mathsf{Q}_l(a,b,c)$ denote the lower triangular $(l+1) \times (l+1)$ matrix in (2.26) with $c \notin \{0, -1, \cdots, -l+1\}$. Then the trace and determinant of Q_l are given respectively by*

$$\operatorname{tr}(\mathsf{Q}_l) = \sum_{j=0}^{l} \frac{(2j-1)!! \prod_{p=j}^{l-1}(c+p)}{(-2)^j \prod_{p=0}^{l-1}(c+p)}, \tag{2.27}$$

and

$$\det(\mathsf{Q}_l) = \frac{\prod_{m=1}^{l}(2m-1)!!}{(-2)^{l(l+1)/2} \prod_{p=0}^{l-1}(c+p)^{l-p}}. \tag{2.28}$$

In particular Q_l is invertible.

Proof. The matrix Q_l has eigenvalues $1, \mathsf{c}_1^1, \ldots, \mathsf{c}_l^l$. Thus $\operatorname{tr}(\mathsf{Q}_l) = 1 + \mathsf{c}_1^1 + \cdots + \mathsf{c}_l^l$ and $\det(\mathsf{Q}_l) = \mathsf{c}_1^1 \times \cdots \times \mathsf{c}_l^l$. Referring to (2.15) the leading hypergeometric coefficients c_m^m can be written as

$$\mathsf{c}_m^m = 2^{-m} \mathsf{b}_m^m \mathsf{d}_{m,m} \prod_{p=0}^{m-1} (c+i)^{-1} = 2^{-m} \mathsf{b}_m^m \prod_{p=0}^{m-1} (c+p)^{-1} \tag{2.29}$$

where in deducing the second equality we have used $\mathsf{d}_{m,m} = 1$. Now from (2.16) we obtain the coefficients b_m^m recursively using the formula

$$\begin{aligned} \mathsf{b}_m^m &= -(m^2 \mathsf{b}_m^{m-1} + (2m-1)\mathsf{b}_{m-1}^{m-1}) \\ &= (-1)^m \prod_{j=1}^{m} (2j-1) = (-1)^m (2m-1)!!, \qquad \forall m \geq 1. \end{aligned} \tag{2.30}$$

Hence by substitution it is seen that the leading hypergeometric coefficients have the explicit form

$$\mathsf{c}_m^m = (-1)^m \frac{(2m-1)!!}{2^m} \prod_{p=0}^{m-1} (c+p)^{-1} \qquad \forall m \geq 1. \tag{2.31}$$

[Note in particular the alternating *sign* of c_m^m affecting both $\mathrm{tr}(\mathsf{Q}_l)$ and $\det(\mathsf{Q}_l)$.] Using this we can write $\mathrm{tr}(\mathsf{Q}_l)$ as

$$\begin{aligned} \mathrm{tr}\,(\mathsf{Q}_l) &= 1 + \sum_{m=1}^{l} \mathsf{c}_m^m = 1 + \sum_{m=1}^{l} (-1)^m \frac{(2m-1)!!2^{-m}}{\prod_{p=0}^{m-1}(c+p)} \\ &= \sum_{m=0}^{l} (-1)^m \frac{(2m-1)!!2^{-m} \prod_{p=m}^{l-1}(c+p)}{\prod_{p=0}^{l-1}(c+p)} \\ &= \frac{2^{-l}\mathsf{F}_l(c)}{\prod_{p=0}^{l-1}(c+p)}, \end{aligned} \tag{2.32}$$

where we have set

$$\mathsf{F}_l(c) = \sum_{m=0}^{l} 2^{l-m} \mathsf{b}_m^m \prod_{p=m}^{l-1} (c+p) = \sum_{m=0}^{l} (-1)^m 2^{l-m} (2m-1)!! \prod_{p=m}^{l-1} (c+p). \tag{2.33}$$

This immediately gives (2.27). Likewise calculating $\det(\mathsf{Q}_l)$ we can write

$$\begin{aligned} \det(\mathsf{Q}_l) &= 1 \times \mathsf{c}_1^1 \cdots \times \mathsf{c}_l^l = \prod_{m=1}^{l} \frac{2^{-m}\mathsf{b}_m^m}{\prod_{p=0}^{m-1}(c+p)} \\ &= \frac{2^{-l(l+1)/2}\mathsf{M}_l}{\prod_{i=0}^{l-1}(c+i)^{l-i}}, \end{aligned} \tag{2.34}$$

where $\mathsf{M}_l = (-1)^{l(l+1)/2} \prod_{m=1}^{l} (2m-1)!!$. This therefore completes the proof. □

Referring to Theorem 2.2.1, specifically, (2.15), (2.16) and recalling (2.9), the first few *leading* hypergeometric coefficients can be seen to be given by

$$\begin{aligned} &\mathsf{c}_1^1(a,b,c) = -\frac{1/2}{c}, \; \mathsf{c}_2^2(a,b,c) = \frac{3/4}{c(c+1)}, \; \mathsf{c}_3^3(a,b,c) = -\frac{15/8}{c(c+1)(c+2)}, \\ &\mathsf{c}_4^4(a,b,c) = \frac{105/16}{c(c+1)(c+2)(c+3)}, \mathsf{c}_5^5(a,b,c) = -\frac{945/32}{c(c+1)(c+2)(c+3)(c+4)}. \end{aligned} \tag{2.35}$$

Using the above we can calculate $\mathsf{M}_1 = -1$, $\mathsf{M}_2 = -3$, $\mathsf{M}_3 = 45$, $\mathsf{M}_4 = 4725$ and $\mathsf{M}_5 = -4465125$ [*see* (2.34)]. Likewise referring to (2.32) we have

$$\begin{aligned}
&\mathsf{F}_1(c) = 2c - 1,\ \mathsf{F}_2(c) = 4c^2 + 2c + 1,\\
&\mathsf{F}_3(c) = 8c^3 + 20c^2 + 10c - 11,\\
&\mathsf{F}_4(c) = 16c^4 + 88c^3 + 140c^2 + 38c + 39,\\
&\mathsf{F}_5(c) = 32c^5 + 304c^4 + 984c^3 + 1196c^2 + 382c - 633.
\end{aligned}$$

The invertibility of the matrix Q_l in (2.26) has the following interesting consequence.

Corollary 2.3.1. *For each polynomial $Q = Q(X)$ of degree $l \geq 1$ there exists a unique even polynomial $\mathsf{P} = \mathsf{P}_d(X)$ with $d = 2l$ such that*

$$\mathbb{T}_d[\mathsf{P}](X) = p_0 + \sum_{m=1}^{l} p_{2m}\mathscr{R}_m(X) = Q(X). \tag{2.36}$$

Proof. Write $Q(X) = b_0 + b_1X + \cdots + b_lX^l$. Then by virtue of (2.26), we have $Q(X) = \langle \mathbf{b}, \mathbf{X}\rangle$ and so (2.36) amounts to $\mathsf{Q}_l\mathbf{P} = \mathbf{b}$ where $\mathbf{P} = (p_{2l}, p_{2l-2}, \cdots, p_0)$ and $\mathbf{b} = (b_l, b_{l-1}, \cdots, b_0)$. The invertibility of Q_l therefore gives $\mathbf{P} = \mathsf{Q}^{-1}\mathbf{b}$ and so the conclusion follows at once upon taking the even polynomial $\mathsf{P}_d(X) = p_0 + p_2X^2 + \cdots + p_{2l}X^{2l}$. More specifically in view of Q_l being lower triangular we have the explicit description

$$\mathbf{P} = \mathsf{Q}_l^{-1}\mathbf{b} \iff \begin{cases} p_{2l} = b_l/\mathsf{c}_l^l,\\ p_{2l-2} = (b_{l-1} - \mathsf{c}_{l-1}^l p_{2l})/\mathsf{c}_{l-1}^{l-1},\\ \ldots\\ p_2 = (b_1 - \sum_{j=2}^{l}\mathsf{c}_1^j p_{2j})/\mathsf{c}_1^1,\\ p_0 = b_0. \end{cases} \tag{2.37}$$

(Note that any polynomial whose even part agrees with P_d will also satisfy (2.36) and that the uniqueness of P_d here is only amongst the *even* polynomials, that is, polynomials with only even powers of X.) □

We now specialise to the compact rank-one symmetric spaces described in Section 2.1 and give an explicit description of the determinant and trace of the matrix Q_l in each case with $n, l \geq 1$ and the parameter c as in Table 2.3. This will then enable us to study the limiting behaviour of these spectral quantities as $n, l \nearrow \infty$.

- ($\mathscr{M} = \mathbb{S}^n$ or $\mathbb{R}\mathbf{P}^n$) With $c = n/2$ the spectrum of the $(l+1)\times(l+1)$ matrix Q_l is given by $\Sigma(\mathsf{Q}_l) = \{1, \mathsf{c}_m^m = (-1)^m(2m-1)!!\prod_{p=0}^{m-1}(n+2p)^{-1} : 1 \leq m \leq l\}$ while the trace and determinant are respectively given by

$$\mathrm{tr}(\mathsf{Q}_l) = \frac{\mathsf{F}_l(n/2)}{\prod_{p=0}^{l-1}(n+2p)} = \frac{\mathsf{F}_l(n/2)}{n(n+2)\dots(n+2l-2)}, \tag{2.38}$$

and

$$\det(\mathsf{Q}_l) = \frac{\mathsf{M}_l}{\prod_{p=0}^{l-1}(n+2p)^{l-p}} = \frac{(-1)^{l(l+1)/2}\prod_{m=1}^{l}(2m-1)!!}{n^l(n+2)^{l-1}\dots(n+2l-2)}. \tag{2.39}$$

In particular $\det(\mathsf{Q}_1) = -1/n$, $\det(\mathsf{Q}_2) = -3/[n^2(n+2)]$ and $\det(\mathsf{Q}_3) = 45/[n^3(n+2)^2(n+4)]$ whilst $\mathsf{F}_1(n/2) = n-1$, $\mathsf{F}_2(n/2) = n^2+n+1$ and $\mathsf{F}_3(n/2) = n^3+5n^2+5n-11$. Note that in the case $n=1$ the spectrum of Q_l is the set $\{(-1)^j : 0 \le j \le l\}$.

- ($\mathscr{M} = \mathbb{C}\mathbf{P}^n$) With $c=n$ the spectrum of Q_l is given by $\Sigma(\mathsf{Q}_l) = \{1, \mathsf{c}_m^m = (-2)^{-m}(2m-1)!!\prod_{p=0}^{m-1}(n+p)^{-1} : 1 \le m \le l\}$ while the trace and determinant are respectively given by

$$\mathrm{tr}(\mathsf{Q}_l) = \frac{\mathsf{F}_l(n)}{2^l\prod_{p=0}^{l-1}(n+p)} = \frac{\mathsf{F}_l(n)}{2^l n(n+1)\dots(n+l-1)}, \tag{2.40}$$

and

$$\det(\mathsf{Q}_l) = \frac{2^{-l(l+1)/2}\mathsf{M}_l}{\prod_{p=0}^{l-1}(n+p)^{l-p}} = \frac{(-2)^{-l(l+1)/2}\prod_{m=1}^{l}(2m-1)!!}{n^l(n+1)^{l-1}\dots(n+l-1)}. \tag{2.41}$$

In particular $\det(\mathsf{Q}_1) = -1/2n$, $\det(\mathsf{Q}_2) = -3/[8n^2(n+1)]$ and $\det(\mathsf{Q}_3) = 45/[64n^3(n+1)^2(n+2)]$ whilst $\mathsf{F}_1(n) = 2n-1$, $\mathsf{F}_2(n) = 4n^2+2n+1$ and $\mathsf{F}_3(n) = 8n^3+20n^2+10n-11$. In the case $n=1$ the spectrum of Q_l is the set $\{1, (-1)^m(2m-1)!!/(2m)!! : 1 \le m \le l\}$ which agrees with that of $\mathbb{S}^2$ and $\mathbb{R}\mathbf{P}^2$ in the previous case.

- ($\mathscr{M} = \mathbb{H}\mathbf{P}^n$) With $c=2n$ the spectrum of Q_l is given by $\Sigma(\mathsf{Q}_l) = \{1, \mathsf{c}_m^m = (-2)^{-m}(2m-1)!!\prod_{p=0}^{m-1}(2n+p)^{-1} : 1 \le m \le l\}$ while the trace and determinant are respectively given by

$$\mathrm{tr}\,(\mathsf{Q}_l) = \frac{\mathsf{F}_l(2n)}{2^l\prod_{p=0}^{l-1}(2n+p)} = \frac{\mathsf{F}_l(2n)}{2^l(2n)(2n+1)(2n+2)\dots(2n+l-1)}, \tag{2.42}$$

and

$$\det(\mathsf{Q}_l) = \frac{2^{-l(l+1)/2}\mathsf{M}_l}{\prod_{p=0}^{l-1}(2n+p)^{l-p}} = \frac{(-2)^{-l(l+1)/2}\prod_{m=1}^{l}(2m-1)!!}{(2n)^l(2n+1)^{l-1}\dots(2n+l-1)}. \tag{2.43}$$

In particular $\det(\mathsf{Q}_1) = -1/4n$, $\det(\mathsf{Q}_2) = -3/[32n^2(2n+1)]$ and $\det(\mathsf{Q}_3) = 45/[64(2n)^3(2n+1)^2(2n+2)]$ whilst $\mathsf{F}_1(2n) = 4n-1$, $\mathsf{F}_2(2n) = 16n^2+4n+1$ and $\mathsf{F}_3(2n) = 64n^3+80n^2+20n-11$. In the case $n=1$ the spectrum of Q_l is the set $\{1, (-1)^m\Gamma(m+1/2)/[\sqrt{\pi}\Gamma(m+2)] : 1 \le m \le l\}$ which agrees with that of $\mathbb{C}\mathbf{P}^2$, $\mathbb{S}^4$ and $\mathbb{R}\mathbf{P}^4$ as described above.

- ($\mathscr{M} = \mathbf{P}^2(\mathrm{Cay})$) With $c = 8$ the determinant is given by

$$\det(\mathsf{Q}_l) = \frac{2^{-l(l+1)/2}\mathsf{M}_l}{\prod_{p=0}^{l-1}(8+p)^{l-p}} = \frac{(-2)^{-l(l+1)/2}\prod_{m=1}^{l}(2m-1)!!}{[8^l \times 9^{l-1} \times \cdots \times (8+l-1)]}, \tag{2.44}$$

and the trace is given by

$$\mathrm{tr}\,(\mathsf{Q}_l) = \frac{2^{-l}\mathsf{F}_l(8)}{\prod_{p=0}^{l-1}(8+p)} = \frac{2^{-l}\mathsf{F}_l(8)}{[(8+l-1)!/7!]}. \tag{2.45}$$

Corollary 2.3.2. *Let $\mathscr{M}$ denote any of the compact rank-one symmetric spaces $\mathbb{S}^n$, $\mathbb{R}\mathbf{P}^n$, $\mathbb{C}\mathbf{P}^n$ or $\mathbb{H}\mathbf{P}^n$. Then for each fixed $l \geq 1$ we have $\lim \mathrm{tr}(\mathsf{Q}_l) = 1$ and $\lim \det(\mathsf{Q}_l) = 0$ as $n \nearrow \infty$.*

Proof. This follows by substituting the relevant value of the parameter c for each of these spaces from Table 2.3 into 2.27-2.28 respectively and passing to the limit. Note that here all the leading hypergeometric coefficients c_m^m (with $1 \leq m \leq l$) converge to zero and so the only eigenvalue of the matrix Q_l with a non-vanishing limit is the eigenvalue one. □

Corollary 2.3.3. *Let $\mathscr{M}$ denote any of the compact rank-one symmetric spaces $\mathbb{S}^n$, $\mathbb{R}\mathbf{P}^n$, $\mathbb{C}\mathbf{P}^n$ or $\mathbb{H}\mathbf{P}^n$. Then with the exception of $\mathbb{S}^1$ and $\mathbb{R}\mathbf{P}^1$, for each fixed $n \geq 1$, we have $\lim \det(\mathsf{Q}_l) = 0$ as $l \nearrow \infty$.*

Proof. First for $\mathscr{M} = \mathbb{S}^n$ or $\mathbb{R}\mathbf{P}^n$ (with $n \geq 1$) by using (2.39) we have

$$\frac{\det(\mathsf{Q}_{l+1})}{\det(\mathsf{Q}_l)} = \frac{(-1)^{l+1}(2l+1)!!}{n(n+2)\cdots(n+2l-2)(n+2l)}. \tag{2.46}$$

For $n = 1$ this quotient is the alternating sequence $(-1)^{l+1}$, thus giving $\det(\mathsf{Q}_l) = (-1)^{(l+2)(l-1)/2}\det(\mathsf{Q}_1) = (-1)^{l(l+1)/2}$ while for $n \geq 3$ odd this gives

$$\frac{\det(\mathsf{Q}_{l+1})}{\det(\mathsf{Q}_l)} = (-1)^{l+1}\frac{(n-2)!!}{(2l+n)!!/(2l+1)!!}. \tag{2.47}$$

Likewise for $n \geq 2$ even we can write

$$\left|\frac{\det(\mathsf{Q}_{l+1})}{\det(\mathsf{Q}_l)}\right| < \frac{1}{2}\frac{(n-2)!!}{(2l+n)!!/(2l+2)!!} = \frac{(n/2-1)!}{2[(l+n/2)!/(l+1)!]}. \tag{2.48}$$

As in either case for fixed $n \geq 2$ and large enough l we have the strict bound $|\det(\mathsf{Q}_{l+1})/\det(\mathsf{Q}_l)| \leq \theta < 1$ the convergence to zero follows at once for $\mathbb{S}^n$ and $\mathbb{R}\mathbf{P}^n$ when $n \geq 2$.

More generally, for the rank-one symmetric spaces $\mathscr{M}$ listed in the statement of the corollary, using (3.15) and upon denoting by c the parameter in Table 2.3, we have

$$\frac{\det(\mathsf{Q}_{l+1})}{\det(\mathsf{Q}_l)} = \frac{(-2)^{-(l+1)}(2l+1)!!}{c(c+1)\cdots(c+l-1)(c+l)}. \tag{2.49}$$

Now, by virtue of $2c$ being an even integer in the remaining cases, we can write

$$\begin{aligned}\left|\frac{\det(\mathsf{Q}_{l+1})}{\det(\mathsf{Q}_l)}\right| &= \frac{(2l+1)!!}{(2l+2c)(2l+2c-2)\cdots(2c+2)(2c)} \\ &< \frac{1/2}{(2l+2c)\cdots(2l+4)}\frac{(2l+2c)!!}{(2l+2c)\cdots(2c+2)(2c)} \\ &\leq \frac{1}{2}\frac{(2c-2)!!}{(2l+2c)\cdots(2l+4)}. \end{aligned} \tag{2.50}$$

This therefore gives the bound

$$\left|\frac{\det(\mathsf{Q}_{l+1})}{\det(\mathsf{Q}_l)}\right| < \frac{1}{2}\frac{(c-1)!(l+1)!}{(l+c)!}, \tag{2.51}$$

and so for each fixed c there exists $p = p(c) \geq 1$ such that for $l \geq p$ we have $|\det(\mathsf{Q}_l)| \leq \theta^{(l-p)}|\det(\mathsf{Q}_p)|$ where $\theta < 1$. This gives the required conclusion. □

Table 2.4: Description of $\mathrm{tr}(\mathsf{Q}_\infty) = F(1/2, 1, c, -1)$ for integer $1 \leq c \leq 5$

c	1	2	3	4	5
$\mathrm{tr}(\mathsf{Q}_\infty)$	$1/\sqrt{2}$	$\frac{2\sqrt{2}+2}{(\sqrt{2}+1)^2}$	$\frac{4(3\sqrt{2}+5)}{3(\sqrt{2}+1)^3}$	$\frac{2(10\sqrt{2}-1)}{5(\sqrt{2}+1)^2}$	$\frac{8(30\sqrt{2}-19)}{35(\sqrt{2}+1)^2}$

Proposition 2.3.1. *Let $\mathsf{Q}_l = \mathsf{Q}_l(a, b, c)$ be as in (2.26). Then for each fixed a, b, c with $c \notin \{0, -1, -2, \dots\}$ we have*

$$\lim_{l\nearrow\infty} \mathrm{tr}(\mathsf{Q}_l) = \sum_{m=0}^{\infty} \frac{(1/2)_m}{(c)_m}(-1)^m = F(1/2, 1; c; z)\Big|_{z=-1}. \tag{2.52}$$

Proof. Referring to (2.27) we have

$$\begin{aligned}\lim_{l\nearrow\infty} \mathrm{tr}(\mathsf{Q}_l) &= 1 + \sum_{m=1}^{\infty} \mathsf{c}_m^m = 1 + \sum_{m=1}^{\infty}(-1)^m \frac{(2m-1)!!2^{-m}}{\prod_{p=0}^{m-1}(c+p)} \\ &= 1 + \sum_{m=1}^{\infty}(-1)^m \frac{\Gamma(m+1/2)\Gamma(c)}{\Gamma(c+m)\Gamma(1/2)} \\ &= \sum_{m=0}^{\infty} \frac{(1/2)_m(1)_m}{m!(c)_m}(-1)^m \end{aligned} \tag{2.53}$$

which is the required conclusion. □

Tables 2.4 and 2.5 give the values of the $\lim \mathrm{tr}(\mathsf{Q}_l)$ for certain values of the parameter c associated with the rank-one symmetric space $\mathscr{M}$ for low dimensions N (*see* Table 2.3).

Table 2.5: Description of $\mathrm{tr}(\mathsf{Q}_\infty)$ for half-integer $1/2 \le c \le 11/2$

c	$1/2$	$3/2$	$5/2$	$7/2$	$9/2$	$11/2$
$\mathrm{tr}(\mathsf{Q}_\infty)$	$1/2$	$\pi/4$	$3(\pi-2)/4$	$15\pi/8-5$	$35\pi/8-77/6$	$315\pi/32-30$

2.4 Differential Operators $\mathscr{L}_\mathsf{P}$ with Zero Action and the Even Polynomials $r_d(X)$

Let us now introduce the functional $\Lambda_d : \mathsf{P} \mapsto \Lambda_d(\mathsf{P}) = \mathbb{T}_d(\mathsf{P})(-ab)$ with $d \ge 2$. Then the action formulated in Theorem 2.2.1 can be characterised using Λ_d as

$$\mathscr{L}_\mathsf{P} F\left(a, b; c; \frac{1-\cos\theta}{2}\right)\Big|_{\theta=0} = \Lambda_d(\mathsf{P}). \tag{2.54}$$

The kernel of Λ_d, denoted $\mathbf{Ker}(\Lambda_d)$, is a d-dimensional hyperplane describing the class of differential operators $\mathscr{L}_\mathsf{P}$ (equivalently polynomials $\mathsf{P} = \mathsf{P}_d(X)$) for which the action (2.54) on the hypergeometric function F vanishes. Now since

$$\mathbf{Ker}(\Lambda_d) = \left\{ \mathsf{P} = \sum_{j=0}^{d} p_j X^j : \mathbb{T}_d[\mathsf{P}](-ab) = p_0 + \sum_{m=1}^{\lfloor d/2 \rfloor} p_{2m} \mathscr{R}_m(-ab) = 0 \right\}, \tag{2.55}$$

denoting by $\mathbb{P}^o_d$ and $\mathbb{P}^e_d$ the orthogonal subspaces of even and odd polynomials of degree at most d with dimensions $1 + \lfloor d/2 \rfloor$ and $\lfloor (d+1)/2 \rfloor$ respectively and writing $\mathscr{R}_1 = \mathscr{R}_1(-ab), \dots \mathscr{R}_{\lfloor d/2 \rfloor} = \mathscr{R}_{\lfloor d/2 \rfloor}(-ab)$ for brevity we have

$$\begin{aligned}
\mathbf{Ker}(\Lambda_d) &= \big\{ \mathsf{P} : \langle (p_0, p_2, \dots, p_{2\lfloor d/2 \rfloor}), (1, \mathscr{R}_1, \dots, \mathscr{R}_{\lfloor d/2 \rfloor}) \rangle = 0 \big\} \\
&= \mathbb{P}^o_d \oplus \{ \mathsf{P} \in \mathbb{P}^e_d : \langle (p_0, p_2, \dots, p_{2\lfloor d/2 \rfloor}), (1, \mathscr{R}_1, \dots, \mathscr{R}_{\lfloor d/2 \rfloor} \rangle = 0 \} \\
&= \mathbb{P}^o_d \oplus r_d(X)^\perp.
\end{aligned} \tag{2.56}$$

Note that in the last line $r_d(X)^\perp \subset \mathbb{P}^e_d$ denotes the orthogonal complement of the vector (here the even polynomial)

$$\begin{aligned}
r_d(X) &= 1 + \sum_{m=1}^{\lfloor d/2 \rfloor} \left\{ \sum_{j=1}^{m} \frac{\mathsf{b}^m_j}{(-2)^j} \frac{(a)_j (b)_j}{(c)_j} \right\} X^{2m} \\
&= 1 + \mathscr{R}_1 X^2 + \cdots + \mathscr{R}_{\lfloor d/2 \rfloor} X^{2\lfloor d/2 \rfloor},
\end{aligned} \tag{2.57}$$

in reference to the subspace $\mathbb{P}^e_d$. Evidently $r_d(X)^\perp$ has dimension $\lfloor d/2 \rfloor$ whilst

$$d = \dim(r_d(X)^\perp) + \dim(\mathbb{P}^o_d) = \lfloor d/2 \rfloor + \lfloor (d+1)/2 \rfloor = \dim(\mathbf{Ker}(\Lambda_d)). \tag{2.58}$$

Specialising now to the compact rank-one symmetric spaces from Section 2.1, by substituting the respective parameters a, b, c from Table 2.3 and the eigenvalues $\lambda^n_k = -ab$

from Table 1, we can obtain specific expressions for the vectors $r_d(X)$ and subsequently $\mathbf{Ker}(\Lambda_d)$, characterising the subspace of differential operators with zero action on the hypergeometric function as formulated in Theorem 2.2.1.

- ($\mathscr{M} = \mathbb{S}^n$) Here $\mathfrak{S}_k(\theta) = \mathscr{P}_k^{((n-2)/2,(n-2)/2)}(\cos\theta) = \mathscr{C}_k^{(n-1)/2}(\cos\theta)$ are the spherical or zonal functions and the polynomial $r_d(X)$ in (2.57) has the coefficients $\mathscr{R}_j = \mathscr{R}_j(k(n+k-1))$ with $1 \le j \le \lfloor d/2 \rfloor$. The first few of these coefficients can be described explicitly as

$$\mathscr{R}_1 = \mathscr{R}_1(\lambda_k^n) = -\frac{k^2+(n-1)k}{n}, \qquad \mathscr{R}_2 = \mathscr{R}_2(\lambda_k^n) = \sum_{i=1}^{4} \frac{\omega_i(n)k^i}{n(n+2)},$$

where $\omega_1 = -2(n-1)^2$, $\omega_2 = 3n^2 - 8n + 5$, $\omega_3 = 6(n-1)$, and $\omega_4 = 3$. Likewise

$$\mathscr{R}_3 = \mathscr{R}_3(\lambda_k^n) = \sum_{i=1}^{6} \frac{\omega_i(n)k^i}{n(n+2)(n+4)},$$

where $\omega_1 = 8(-2n^3+5n^2-4n+1)$, $\omega_2 = 2(15n^3-53n^2+57n-19)$, $\omega_3 = 15(-n^3+7n^2-11n+5)$, $\omega_4 = -15(3n^2-8n+5)$, $\omega_5 = -45(n-1)$, and $\omega_6 = -15$.

- ($\mathscr{M} = \mathbb{R}\mathbf{P}^n$) Here $\mathfrak{S}_k(\theta) = \mathscr{P}_{2k}^{((n-2)/2,(n-2)/2)}(\cos\theta) = \mathscr{C}_{2k}^{(n-1)/2}(\cos\theta)$ are the spherical or zonal functions and the polynomial $r_d(X)$ in (2.57) has the coefficients $\mathscr{R}_j = \mathscr{R}_j(2k(n+2k-1))$ with $1 \le j \le \lfloor d/2 \rfloor$. The first few of these coefficients can be described as

$$\mathscr{R}_1 = \mathscr{R}_1(\lambda_k^n) = -\frac{4k^2+2(n-1)k}{n}, \qquad \mathscr{R}_2 = \mathscr{R}_2(\lambda_k^n) = \sum_{i=1}^{4} \frac{\omega_i(n)k^i}{n(n+2)},$$

where $\omega_1 = -4(n-1)^2$, $\omega_2 = 4(3n^2-8n+5)$, $\omega_3 = 48(n-1)$, and $\omega_4 = 48$. Likewise

$$\mathscr{R}_3(\lambda_k^n) = \sum_{i=1}^{6} \frac{\omega_i(n)k^i}{n(n+2)(n+4)},$$

where $\omega_1 = 16(-2n^3+5n^2-4n+1)$, $\omega_2 = 8(15n^3-53n^2+57n-19)$, $\omega_3 = -120(n^3-7n^2+11n-5)$, $\omega_4 = -240(3n^2-8n+5)$, $\omega_5 = -1440(n-1)$, and $\omega_6 = -960$.

- ($\mathscr{M} = \mathbb{C}\mathbf{P}^n$) Here $\mathfrak{S}_k(\theta) = \mathscr{P}_k^{(n-1,0)}(\cos\theta)$ are the spherical functions and the polynomial $r_d(X)$ in (2.57) has the coefficients $\mathscr{R}_j = \mathscr{R}_j(k(n+k))$ with $1 \le j \le \lfloor d/2 \rfloor$. The first few coefficients can be described as

$$\mathscr{R}_1 = \mathscr{R}_1(\lambda_k^n) = -\frac{k^2+kn}{2n}, \qquad \mathscr{R}_2 = \mathscr{R}_2(\lambda_k^n) = \sum_{i=1}^{4} \frac{\omega_i(n)k^i}{4n(n+1)},$$

where $\omega_1 = -n(n+1), \omega_2 = (3n^2-n-1), \omega_3 = 6n$, and $\omega_4 = 3$. Likewise for $\mathscr{R}_3(\lambda_k^n)$ we can write

$$\mathscr{R}_3 = \mathscr{R}_3(\lambda_k^n) = \sum_{i=1}^{6} \frac{\omega_i(n)k^i}{8n(n+1)(n+2)},$$

where here we have $\omega_1 = -4n(n^2+3n+2)$, $\omega_2 = 15n^3+11n^2-12n-8$, $\omega_3 = -15n(n^2-2n-2)$, $\omega_4 = -15(3n^2-n-1)$, $\omega_5 = -45n$ and $\omega_6 = -15$.

- ($\mathscr{M} = \mathbb{H}\mathbf{P}^n$) Here $\mathfrak{S}_k(\theta) = \mathscr{P}_k^{(2n-1,1)}(\cos\theta)$ are the spherical functions and the polynomial $r_d(X)$ in (2.57) has $\mathscr{R}_j = \mathscr{R}_j(k(2n+k+1))$ with $1 \le j \le \lfloor d/2 \rfloor$ as coefficients. The first few of these can be described as

$$\mathscr{R}_1 = \mathscr{R}_1(\lambda_k^n) = -\frac{k^2+(2n+1)k}{4n}, \qquad \mathscr{R}_2 = \mathscr{R}_2(\lambda_k^n) = \sum_{i=1}^{4} \frac{\omega_i(n)k^i}{8n(2n+1)},$$

  where $\omega_1 = -2(2n^2+5n+2), \omega_2 = 12n^2+10n-1, \omega_3 = 6(2n+1)$, and $\omega_4 = 3$. Likewise for $\mathscr{R}_3(\lambda_k^n)$ we have

$$\mathscr{R}_3 = \mathscr{R}_3(\lambda_k^n) = \sum_{i=1}^{6} \frac{\omega_i(n)k^i}{16n(2n+1)(2n+2)},$$

  where $\omega_1 = -2(16n^3+92n^2+110n+34)$, $\omega_2 = 2(60n^3+172n^2+93n-4)$, $\omega_3 = -15(8n^3+4n^2-14n-7)$, $\omega_4 = -15(12n^2+10n-1)$, $\omega_5 = -45(2n+1)$, and $\omega_6 = -15$.

- ($\mathscr{M} = \mathbf{P}^2(\mathrm{Cay})$) In this case $\mathfrak{S}_k(\theta) = \mathscr{P}_k^{(7,3)}(\cos\theta)$ represent the spherical functions and the polynomial $r_d(X)$ in (2.57) has the coefficients given by $\mathscr{R}_j = \mathscr{R}_j(k(k+11))$ with $1 \le j \le \lfloor d/2 \rfloor$. The first few of these are $\mathscr{R}_1(\lambda_k^n) = -(k^2+11k)/16$, $\mathscr{R}_2(\lambda_k^n) = (k^4+22k^3+115k^2-66k)/96$ and $\mathscr{R}_3(\lambda_k^n) = -(k^6+33k^5+345k^4+935k^3-2082k^2+1056k)/384$.

2.5 Extension of the Results to the Generalised Hypergeometric Functions

The generalised hypergeometric function ${}_pF_q(\mathbf{a};\mathbf{b};z)$ where $\mathbf{a} = (a_1,\dots,a_p)$ and $\mathbf{b} = (b_1,\dots,b_q)$ is defined by the series

$${}_pF_q(\mathbf{a};\mathbf{b};z) = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k\dots(a_p)_k}{(b_1)_k(b_2)_k\dots(b_q)_k}\frac{z^k}{k!}, \tag{2.59}$$

that converges for all finite values of z when $p \le q$ and all $|z| < 1$ when $p = q+1$. The series diverges for all non-zero z when $p > q+1$. In the case $p = q+1$ the series converges

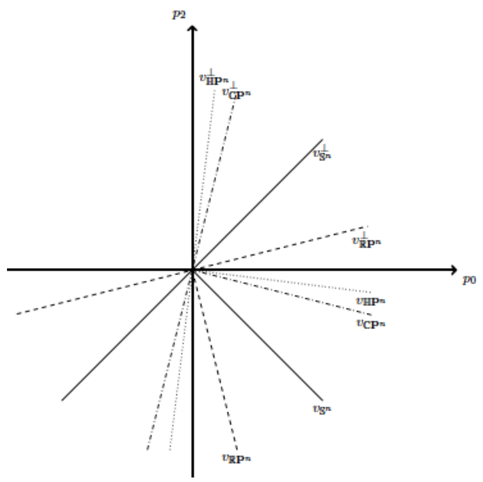

Figure 2.1: Illustration of the space of zero action differential operators $\mathscr{L}_{\mathsf{P}}$ for $d=2$ in the (p_0,p_2)-plane. Here $\mathbf{Ker}(\Lambda_2)=\mathbb{P}_2^o \oplus \{(p_0,p_2): p_0+p_2\mathscr{R}_1=0\}$ where up to a scalar factor $v_{\mathbb{S}^n}=(1,-[k^2+(n-1)k]/n)$, $v_{\mathbb{R}\mathbf{P}^n}=(1,-[4k^2+(2n-2)k]/n)$, $v_{\mathbb{C}\mathbf{P}^n}=(1,-[k^2+kn]/(2n))$ and $v_{\mathbb{H}\mathbf{P}^n}=(1,-[k^2+(2n+1)k]/(4n))$.

absolutely for all $|z|=1$ if $\Re(\sum_i b_i-\sum_i a_i)>0$ and converges conditionally for all $|z|=1$ and $z\neq 1$ if $-1<\Re(\sum_i b_i-\sum_i a_i)\leq 0$ while the series diverges if $\Re(\sum_i b_i-\sum_i a_i)\leq -1$. Clearly when any of the parameters a_i (with $1\leq i\leq p$) is a non-positive integer the series terminates and becomes a polynomial in z.

Indeed, here, for $m\geq 1$, we have the differential identity

$$\frac{d^m}{dz^m}{}_pF_q(\mathbf{a};\mathbf{b};z)=\frac{\prod_{i=1}^p(a_i)_m}{\prod_{j=1}^q(b_j)_m}\sum_{k=0}^{\infty}\frac{(a_1+m)_k(a_2+m)_k\dots(a_p+m)_k}{(b_1+m)_k(b_2+m)_k\dots(b_q+m)_k}\frac{z^k}{k!}. \tag{2.60}$$

The statement of Lemma 2.2.1 in this context becomes a product of p Pochhammer symbols,

$$(a_1)_j(a_2)_j\dots(a_p)_j=\prod_{k=0}^{j-1}\prod_{i=1}^{p}(a_i+k)=\sum_{l=0}^{j}\mathsf{d}_{l,j}(\mathbf{a})\left[\prod_{i=1}^{p}a_i\right]^l. \tag{2.61}$$

Here the scalars $\mathsf{d}_{l,j}(\mathbf{a})$ are the coefficients of the 'eigenvalue' $X=\prod_{i=1}^p a_i$ in the polynomial expansion of the product on the left in X. With the operator $\mathscr{L}_{\mathsf{P}}=\mathsf{P}_d(d/d\theta)$ as in (2.12) we can then state the following theorem.

Theorem 2.5.1. *Let $\mathscr{L}_{\mathsf{P}}$ be the differential operator as defined in* (2.12). *Then for $|z|<1$, and $\mathbf{a}=(a_1,\dots,a_p)$ and $\mathbf{b}=(b_1,\dots,b_q)$, the generalised hypergeometric function $F(\mathbf{a};\mathbf{b};z)$ satisfies the differential identity*

$$\mathscr{L}_{\mathsf{P}}\left[{}_pF_q\right]\left(\mathbf{a};\mathbf{b};\frac{1-\cos\theta}{2}\right)\Bigg|_{\theta=0}=p_0+\sum_{m=1}^{\lfloor d/2\rfloor}p_{2m}\sum_{j=1}^{m}\mathsf{c}_j^m(\mathbf{a},\mathbf{b})\left[-\prod_{i=1}^{p}a_i\right]^j, \tag{2.62}$$

where the scalars $\mathsf{c}_j^m(\mathbf{a},\mathbf{b})$ *are called the generalised hypergeometric coefficients, and are explicitly given by*

$$\mathsf{c}_j^m(\mathbf{a},\mathbf{b}) = \sum_{i=j}^{m} \frac{(-1)^{i+j}\mathsf{b}_i^m \mathsf{d}_{j,i}}{2^i \prod_{l=1}^{q}(b_l)_i}. \tag{2.63}$$

Here the scalars b_i^m *are defined in* (2.82)*, and the scalars* $\mathsf{d}_{j,i} = \mathsf{d}_{j,i}(\mathbf{a})$ *are defined in* (2.61).

Having the above theorem at our disposal we can move forward, and again as in the case $p=2$ and $q=1$, write the differential action (2.62) in the form

$$\mathscr{L}_{\mathsf{P}}\left[{}_pF_q\right]\left(\mathbf{a};\mathbf{b};\frac{1-\cos\theta}{2}\right)\Bigg|_{\theta=0} = p_0 + \sum_{m=1}^{\lfloor d/2\rfloor} p_{2m}\mathscr{R}_m\left(-\prod_{i=1}^{p} a_i\right), \tag{2.64}$$

where $\mathscr{R}_m(X) = \mathscr{R}_m(X;\mathbf{a},\mathbf{b}) = \sum_{j=1}^{m}\mathsf{c}_j^m(\mathbf{a},\mathbf{b})X^j$ and the generalised hypergeometric coefficients c_j^m are as in (2.63). Now upon introducing the operator

$$\mathbb{T}_d : \mathsf{P}_d(X) = p_0 + \sum_{j=1}^{d} p_j X^j \mapsto Q(X) = p_0 + \sum_{m=1}^{\lfloor d/2\rfloor} p_{2m}\mathscr{R}_m(X), \tag{2.65}$$

it is readily seen that the RHS of (2.62)-(2.64) is $Q(-\prod_{i=1}^{p} a_i) = \mathbb{T}_d[\mathsf{P}](-\prod_{i=1}^{p} a_i)$. The operator $\mathbb{T}_d$ here admits the matrix representation (with $l = \lfloor d/2\rfloor$, $l \geq 1$) $\mathbb{T}_d[\mathsf{P}] = \langle \mathsf{Q}_l\mathbf{P},\mathbf{X}\rangle$ where Q_l is the lower triangular matrix in (2.26) [with entries $\mathsf{c}_j^m(\mathbf{a},\mathbf{b})$ in place of $\mathsf{c}_j^m(a,b,c)$], $\mathbf{P} = (p_{2l},\cdots,p_2,p_0)$ is the vector of the *even* coefficients of P_d and $\mathbf{X} = (X^l,\cdots,X,1)$. A basic inspection shows that the eigenvalues of Q_l are the leading hypergeometric coefficients $\mathsf{c}_l^l, \mathsf{c}_{l-1}^{l-1},\cdots,\mathsf{c}_1^1, 1$.

Corollary 2.5.1. *Let* $\mathsf{Q}_l = \mathsf{Q}_l(\mathbf{a},\mathbf{b})$ *be as in* (3.77)*. Then for each fixed* $\mathbf{a},\mathbf{b}$ *with* $b_j \notin \{0,-1,-2,\dots\}$ *for all* $1 \leq j \leq q$ *we have*

$$\lim_{l\nearrow\infty}\operatorname{tr}(\mathsf{Q}_l) = \sum_{k=0}^{\infty}\frac{(1/2)_k(1)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\frac{(-1)^k}{k!} = F((1/2,1);\mathbf{b};z)\Big|_{z=-1}, \tag{2.66}$$

whilst $\lim\det\mathsf{Q}_l = 0$ *as* $l\nearrow\infty$.

2.6 The Hypergeometric Function ${}_2F_1 = F(a,b,c;z)$

The hypergeometric function is defined on the unit disk $\{z\in\mathbb{C} : |z|<1\}$ by the infinite series

$$F(a,b,c;z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \tag{2.67}$$

with $a,b,c\in\mathbb{C}$ and $c\neq 0,-1,-2,\cdots$ Here $(x)_m$ denotes the Pochhammer symbol or the rising factorial defined by

$$(x)_m = \prod_{p=0}^{m-1}(x+p) = \frac{\Gamma(x+m)}{\Gamma(x)}, \tag{2.68}$$

where the second equality assumes x and $x+m$ are not negative integers or zero. The hypergeometric function admits an analytic continuation beyond its circle of convergence along any curve avoiding the points $z=1$ and infinity, in fact, for $\Re c > \Re b > 0$ by Euler's integral representation formula we have

$$F(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt, \tag{2.69}$$

and for all z in $\mathbb{C}$ cut along the real axis from $z=1$ to ∞. The hypergeometric function satisfies the differential identity

$$\frac{d}{dz}F(a,b;c;z) = \frac{ab}{c}F(a+1,b+1;c+1;z), \tag{2.70}$$

from which one can easily derive the analogue for m derivatives

$$\frac{d^m}{dz^m}F(a,b;c;z) = \frac{(a)_m(b)_m}{(c)_m}F(a+m,b+m;c+m;z). \tag{2.71}$$

Note that from (2.67) we have the identity $F(a,b;c;0)=1$ which was used in the proof of Theorem 2.2.1. The hypergeometric function arises as a solution to the hypergeometric differential equation

$$z(1-z)\frac{d^2w}{dz^2} + (c-(a+b+1)\,z)\,\frac{dw}{dz} - abw = 0. \tag{2.72}$$

This equation can be reached from any second-order ordinary differential equation with at most three regular singular points by a suitable change of variables. By the change of variables $z=(1-t)/2$, and setting $a=-k, b=\alpha+\beta+k+1$, and $c=\alpha+1$, one can transform (2.72) into the well known Jacobi differential equation,

$$(1-t^2)\frac{d^2w}{dt^2} + (\beta-\alpha-(\alpha+\beta+2)t)\frac{dw}{dt} + k(\alpha+\beta+k+1)w = 0, \tag{2.73}$$

which is solved by the Jacobi polynomial $w = \mathscr{P}_k^{(\alpha,\beta)}(t)$, a special case of the hypergeometric function. As a matter of fact some important special cases of the hypergeometric function $F(a,b;c;z)$ for future reference are:

- The Legendre polynomial $P_k(t)$, $k \geq 0$,

$$\begin{aligned} P_k(t) &= F(-k,k+1;1;(1-t)/2) \\ &= \frac{1}{2^k k!}\frac{d^k}{dt^k}\left[\left(t^2-1\right)^k\right]. \end{aligned} \tag{2.74}$$

- The Gegenbauer polynomial $\mathscr{C}_k^\nu(t)$, $\nu > -1/2$, $k \geq 0$,

$$\begin{aligned} \mathscr{C}_k^\nu(t) &= F(-k,2\nu+k;\nu+1/2;(1-t)/2) \\ &= \frac{(-1)^k}{2^k(\nu+1/2)_k}\left(1-t^2\right)^{-\nu+1/2}\frac{d^k}{dt^k}\left[\left(1-t^2\right)^{k+\nu-1/2}\right]. \end{aligned} \tag{2.75}$$

- The Jacobi polynomial $\mathscr{P}_k^{(\alpha,\beta)}(t)$, $k \geq 0$, $\alpha, \beta > -1$,

$$\begin{aligned}\mathscr{P}_k^{(\alpha,\beta)}(t) &= F(-k, \alpha+\beta+k+1; \alpha+1; (1-t)/2) \\ &= \frac{(-1)^k}{2^k(\alpha+1)_k}(1-t)^{-\alpha}(1+t)^{-\beta}\frac{d^k}{dt^k}\left[(1-t)^{\alpha}(1+t)^{\beta}(1-t^2)^k\right].\end{aligned} \tag{2.76}$$

Note in particular that in the Gegenbauer and Jacobi cases we have $\mathscr{P}_k^{(\alpha,\beta)}(1) = 1$ and $\mathscr{C}_k^{\nu}(1) = 1$ by the choice of normalization.

2.7 The Bell Polynomial $\mathsf{B}_{m,j}$ and Faà di Bruno's Formula

To prove the main theorem of Section 2.2 we make use of Faà di Bruno's formula, a generalised chain rule, in order to write derivatives of $F(a, b; c; z)$ in terms of (incomplete) Bell polynomials $\mathsf{B}_{m,j}(x)$. These are defined for $x = (x_1, \ldots, x_{m-j+1})$ as

$$\mathsf{B}_{m,j}(x) = \sum \frac{m!}{k_1!k_2!\ldots k_{m-j+1}!}\prod_{i=1}^{m-j+1}\left(\frac{x_i}{i!}\right)^{k_i} \tag{2.77}$$

where the sum is taken over all sequences of non-negative integers $k_1, \ldots, k_{m-j+1}$ such that

$$k_1 + \cdots + j_{m-j+1} = j, \text{ and } k_1 + 2k_2 + \cdots + (m-j+1)k_{m-j+1} = m. \tag{2.78}$$

For smooth functions f, g, Faà di Bruno's formula then asserts that

$$\frac{d^m}{dx^m}f(g(x)) = \sum_{j=1}^{m} f^{(j)}(g(x)) \cdot \mathsf{B}_{m,j}\left(g'(x), g''(x), \ldots, g^{(m-j+1)}(x)\right). \tag{2.79}$$

Setting $g(x) = \cos x$ in Faà di Bruno's formula and evaluating at $x = 0$ results in the special case

$$\left.\frac{d^{2m}}{dx^{2m}}f(\cos x)\right|_{x=0} = \left.\left\{\sum_{j=1}^{m}\mathsf{b}_j^m\frac{d^j}{dt^j}f(t)\right\}\right|_{t=1}, \tag{2.80}$$

where the coefficients

$$\mathsf{b}_j^m = \mathsf{B}_{2m,j}(-\sin x, -\cos x, \sin x, \ldots)\Big|_{x=0} \tag{2.81}$$

satisfy the recursive formula, for $m, j > 0$ integers

$$\mathsf{b}_j^m = \begin{cases}(-1)^m & \text{if } j = 1 \\ -\left(j^2\mathsf{b}_j^{m-1} + (2j-1)\mathsf{b}_{j-1}^{m-1}\right) & \text{if } j < m+1 \\ 0 & \text{if } j > m.\end{cases} \tag{2.82}$$

Here we have taken advantage of the fact that $\mathsf{B}_{2l,j}(0,x_2,x_3,\ldots,x_{m-j+1}) = 0$ for all $j \geq l+1 > 0$. This can be seen by setting $j \geq l+1$, $k_1 = 0$, and taking non-negative integers $0 = k_1,\ldots,k_{2l-j+1}$ which satisfy (2.78) with $m = 2l$. Then clearly $\sum_{i=2}^{2l-j+1} k_i = j$, but the second condition in (2.78) gives

$$\sum_{i=2}^{2l-j+1} ik_i = \sum_{i=2}^{2l-j+1} (i-2)k_i + 2\sum_{i=2}^{2l-j+1} k_i \geq 2(l+1) > 2l, \tag{2.83}$$

which is a contradiction. So we must have $k_1 \neq 0$, and hence the terms of $\mathsf{B}_{2l,j}$ depend on k_1.

2.8 The Polynomials $\mathscr{R}_m$, $4 \leq m \leq 5$

In this appendix we present the hypergeometric coefficients $\mathsf{c}_j^m = \mathsf{c}_j^m(a,b,c)$ and the polynomials $\mathscr{R}_m(X) = \mathscr{R}_m(X;a,b,c)$ for $m = 4,5$ using the formulation and explicit description provided in Theorem 2.2.1. Larger values of m are also possible and follow a similar patter but they become increasingly more complex. Indeed starting from $m = 4$, we can write

$$\mathscr{R}_4(X;a,b,c) = \sum_{l=1}^{4} \mathsf{c}_l^4 X^l = \sum_{l=1}^{4} \frac{\mathsf{q}_l^4(a,b,c)}{2^4 \mathscr{A}_4(c)} X^l, \tag{2.84}$$

where $\mathscr{A}_4(c) = c(c+1)(c+2)(c+3)$, and then

$$\begin{aligned}
\mathsf{q}_1^4 =& -252c^2(a+b) + 1260c(a+b) - 630(a+b)^3 - 882(a+b)\\
&- 1260(a+b)^2 + 840c(a+b)^2 + 8c^3 - 204c^2 + 508c - 204,\\
\mathsf{q}_2^4 =& 1155(a+b)^2 - 1260c(a+b) + 1260(a+b) + 252c^2 - 840c + 357,\\
\mathsf{q}_3^4 =& 420c - 210 - 630(a+b),\\
\mathsf{q}_4^4 =& 105.
\end{aligned}$$

In a similar way for $m = 5$ we can write

$$\mathscr{R}_5(X;a,b,c) = \sum_{l=1}^{5} \mathsf{c}_l^5 X^l = \sum_{l=1}^{5} \frac{\mathsf{q}_l^5(a,b,c)}{2^5 \mathscr{A}_5(c)} X^l, \tag{2.85}$$

where $\mathscr{A}_5(c) = c(c+1)(c+2)(c+3)(c+4)$, and then a set of direct but lengthy calculations give

$$
\begin{aligned}
\mathsf{q}_1^5 = &- 22680(a+b)^4 + 37800c(a+b)^3 - 75600(a+b)^3 - 98280(a+b)^2 \\
&+ 103320c(a+b)^2 - 17640c^2(a+b)^2 + 2040c^3(a+b) \\
&- 34560c^2(a+b) + 98400c(a+b) - 56880(a+b) - 16c^4 + 1880c^3 \\
&- 17480c^2 + 32080c - 11904, \\
\mathsf{q}_2^5 = &47250(a+b)^3 + 104580(a+b)^2 - 69300c(a+b)^2 \\
&+ 26460c^2(a+b) - 117180c(a+b) + 81270(a+b) - 2040c^3 \\
&+ 25740c^2 - 53040c + 20340, \\
\mathsf{q}_3^5 = &- 33075(a+b)^2 + 37800c(a+b) - 37800(a+b) - 8820c^2 \\
&+ 26460c - 11025, \\
\mathsf{q}_4^5 = &- 6300c + 3150 + 9450(a+b), \\
\mathsf{q}_5^5 = &- 945.
\end{aligned}
$$

Chapter 3

Extensions of Differential Actions $\mathscr{L}_{\mathsf{P}}$ and Hypergeometric Series to Variables X_1 and X_2

3.1 Jacobi Analogue of the Hypergeometric Series

An analogue of theorem 2.2.1 exists in terms of Jacobi polynomials.

Theorem 3.1.1. *Let $\mathsf{P}_d(X) = p_0 + \sum_{1\le i\le d} p_i X^i$ be a polynomial of degree $d \geq 2$ and let $\mathscr{L}_{\mathsf{P}}$ denote the differential operator defined as*

$$\mathscr{L}_{\mathsf{P}} = \mathsf{P}_d(d/d\theta) = p_0 + \sum_{i=1}^{d} p_i d^i/d\theta^i. \tag{3.1}$$

Then for $\alpha, \beta > -1$, the action of $\mathscr{L}_{\mathsf{P}}$ satisfies the relation

$$p_0 + \sum_{l=1}^{\lfloor d/2 \rfloor} p_{2l} \mathscr{R}_l(-ab). \tag{3.2}$$

Here, we have

$$\mathscr{R}_l = \sum_{j=1}^{l} c_j^l(\alpha, \beta) X^j \tag{3.3}$$

with coefficients c_j^l, $1 \leq j \leq m$ (hereafter called the Jacobi coefficients) given by

$$c_j^l(\alpha, \beta) = \sum_{m=j}^{l} \mathsf{a}_m^l \mathsf{b}_j^m. \tag{3.4}$$

Furthermore, the coefficients b_j^m are given by $\mathsf{b}_m^m = 1$, $\mathsf{b}_1^{m+1} = -m(m+\alpha+\beta+1)\mathsf{b}_1^m$ for $m \geq 1$, and $\mathsf{b}_j^{m+1} = \mathsf{b}_{j-1}^m - m(m+\alpha+\beta+1)\mathsf{b}_j^m$ for $2 \leq j \leq m$. The coefficients a_m^l are *given by*

$$\mathsf{a}_m^l = \frac{2^{-m}\Gamma(\alpha+1)}{\Gamma(\alpha+m+1)} B_{2l,m}(0,-1,0,+1,0,\dots) \tag{3.5}$$

where $B_{k,m}$ are the partial exponential Bell polynomials.

Table 3.1: Spectral data for rank-one symmetric spaces with dimension N

$\mathscr{M}$	N	λ_k^n	$M_k^n(\mathscr{M})$	$\mathrm{Vol}(\mathscr{M})$
$\mathbb{S}^n$	n	$k(k+n-1)$	$\frac{(2k+n-1)(k+n-2)!}{k!(n-1)!}$	$\frac{2\pi^{(n+1)/2}}{\Gamma((n+1)/2)}$
$\mathbb{R}\mathbf{P}^n$	n	$2k(2k+n-1)$	$\frac{(4k+n-1)(2k+n-2)!}{(2k)!(n-1)!}$	$\frac{\pi^{(n+1)/2}}{\Gamma((n+1)/2)}$
$\mathbb{C}\mathbf{P}^n$	$2n$	$k(k+n)$	$\frac{2k+n}{n}\left(\frac{\Gamma(k+n)}{\Gamma(n)k!}\right)^2$	$\frac{4^n\pi^n}{n!}$
$\mathbb{H}\mathbf{P}^n$	$4n$	$k(k+2n+1)$	$\frac{(2k+2n+1)(k+2n)}{(2n)(2n+1)(k+1)}\left(\frac{\Gamma(k+2n)}{k!\Gamma(2n)}\right)^2$	$\frac{4^{2n}\pi^{2n}}{(2n+1)!}$
$\mathbf{P}^2(\mathrm{Cay})$	16	$k(k+11)$	$6(2k+11)\frac{\Gamma(k+8)\Gamma(k+11)}{7!11!k!\Gamma(k+4)}$	$\frac{3!(4\pi)^8}{11!}$

3.2 First Three Jacobi Polynomials

The first three Jacobi polynomials are given as follows.

$$\mathscr{R}_1(X) = -\frac{X}{2(\alpha+1)},\ \mathscr{R}_2(X) = \frac{-(\alpha+2+3\beta)X^1+3X^2}{4(\alpha+1)(\alpha+2)} \tag{3.6}$$

$$\mathscr{R}_3(X) = \frac{-15X^3+(\alpha+3\beta+2)X^2-(4\alpha^2+30\alpha\beta+30\beta^2+20\alpha+60\beta+24)X}{8(\alpha+1)(\alpha+2)(\alpha+3)} \tag{3.7}$$

These are obtained by substituting $a=-k, b=k+\alpha+\beta+1, c=\alpha+1$ into the first three hypergeometric series.

3.3 Jacobi Equivalent of the Matrix

Definition 3.3.1. *For α,β as above and $d\geq 2$ we denote by $\mathbb{T}=\mathbb{T}_d$ the operator acting on the space of polynomials P_d as in Theorem 3.1.1 defined by*

$$\mathbb{T}_d : \mathsf{P}_d(X) = p_0 + \sum_{j=1}^{d} p_j X^j \mapsto Q_{\lfloor d/2\rfloor}(X) = p_0 + \sum_{l=1}^{\lfloor d/2\rfloor} p_{2l}\mathscr{R}_l(X), \tag{3.8}$$

where $\mathscr{R}_l = \mathscr{R}_m(X;\alpha,\beta)$ are the polynomials defined in Theorem 3.1.1.

Note that with the aid of the above definition the conclusion of Theorem 3.1.1 can be rewritten in the form

$$\left(\mathscr{L}_{\mathsf{P}}F\right)(\alpha,\beta) = \mathbb{T}_d[\mathsf{P};\alpha,\beta](-ab) = Q_{\lfloor d/2\rfloor}(-ab). \tag{3.9}$$

For the sake of clarity and to fix ideas let us pause briefly to look into this in more detail for certain smaller values of d. Indeed using the formulation of the polynomials $\mathscr{R}_m$ in Theorem 3.1.1 and the explicit description for $1 \le m \le 3$ in (3.6) it is seen that for $2 \le d \le 3$ we have $Q(X) = \mathbb{T}_d[\mathsf{P}](X) = p_0 + p_2\mathscr{R}_1(X) = p_0 + p_2\mathsf{c}_1^1 X = p_0 - p_2 X/2(\alpha+1)$; in particular $Q(-ab) = p_0 + p_2 ab/2(\alpha+1)$. Likewise for $4 \le d \le 5$ and again with $Q(X) = \mathbb{T}_d[\mathsf{P}](X)$ we have

$$\begin{aligned} Q(X) &= p_0 + p_2\mathscr{R}_1 + p_4\mathscr{R}_2 = p_0 + p_2\mathsf{c}_1^1 X + p_4(\mathsf{c}_2^2 X^2 + \mathsf{c}_1^2 X) \\ &= p_0 + (p_4\mathsf{c}_1^2 + p_2\mathsf{c}_1^1)X + p_4\mathsf{c}_2^2 X^2. \end{aligned} \tag{3.10}$$

Returning now to (3.8) and the description of the polynomial $Q = \mathbb{T}_d[\mathsf{P}]$ it is evident that

$$\begin{aligned} \mathbb{T}_d[\mathsf{P}] = Q(X) &= p_0 + \sum_{l=1}^{\lfloor d/2\rfloor} p_{2l}\mathscr{R}_l(X) \\ &= p_0 + \sum_{l=1}^{\lfloor d/2\rfloor}\sum_{j=1}^{l} p_{2l}c_j^l X^j \\ &= p_0 + \sum_{j=1}^{\lfloor d/2\rfloor}\left[\sum_{l=j}^{\lfloor d/2\rfloor} p_{2l}c_j^l\right] X^j. \end{aligned} \tag{3.11}$$

Hence setting $m = \lfloor d/2\rfloor$ with $m \ge 1$, the latter, upon introducing the $(m+1)\times(m+1)$ lower triangular matrix Q_m can be written as $Q(X) = \langle \mathsf{Q}_m\mathbf{P}, \mathbf{X}\rangle$, that is,

$$Q(X) = \left\langle \begin{bmatrix} c_m^m & 0 & 0 & \cdots & 0 \\ c_{m-1}^m & c_{m-1}^{m-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_1^m & c_1^{m-1} & \cdots & c_1^1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} p_{2m} \\ p_{2(m-1)} \\ \vdots \\ p_2 \\ p_0 \end{bmatrix}, \begin{bmatrix} X^m \\ X^{m-1} \\ \vdots \\ X \\ 1 \end{bmatrix} \right\rangle. \tag{3.12}$$

Here $\mathbf{P} = (p_{2m}, \cdots, p_2, p_0)$ is the vector of the *even* coefficients of P_d while $\mathbf{X} = (X^m, \cdots, X, 1)$. Moreover referring to the matrix Q_m it is seen that the eigenvalues are given by the leading hypergeometric coefficients $c_m^m, c_{m-1}^{m-1}, \cdots, c_1^1, 1$. These are the eigenvalues, the trace is $1 + c_1^1 + \cdots + c_m^m$, and the determinant is given by $c_1^1 \times \ldots c_m^m$.

3.4 Infinite Trace Limit of α and β

Proposition 3.4.1. *Let $\mathsf{Q}_m = \mathsf{Q}_m(\alpha, \beta)$ be as in* (2.26)*. Then for each fixed α, β we have*

$$\lim_{l \nearrow \infty} \operatorname{tr}(\mathsf{Q}_m) = \sum_{l=0}^{\infty} \frac{(1/2)_l}{(\alpha+1)_l}(-1)^l = F(1/2, 1; \alpha+1; z)\Big|_{z=-1}. \tag{3.13}$$

3.5 Trace and Determinant in Terms of α and β

Theorem 3.5.1. *Let $\mathsf{Q}_m = \mathsf{Q}_m(\alpha, \beta)$ denote the lower triangular $(m+1) \times (m+1)$ matrix in* (3.77)*. Then the trace and determinant of Q_l are given respectively by*

$$\operatorname{tr}(\mathsf{Q}_m) = \sum_{j=0}^{m} \frac{(2j-1)!! \prod_{p=j}^{m-1}(\alpha+1+p)}{(-2)^j \prod_{p=0}^{m-1}(\alpha+1+p)}, \tag{3.14}$$

and

$$\det(\mathsf{Q}_m) = \frac{\prod_{l=1}^{m}(2l-1)!!}{(-2)^{m(m+1)/2} \prod_{p=0}^{m-1}(\alpha+1+p)^{l-p}}. \tag{3.15}$$

In particular Q_m is invertible.

3.6 $f(\theta)\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)$ Derivative Expansions

The interest in the expansion of the identity

$$\frac{d^m}{d\theta^m}\left(f(\theta)\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\right)\Big|_{\theta=0}, \ m \geq 1 \tag{3.16}$$

can be expressed in the following proposition.

Proposition 3.6.1. *The expansions above can be summarised via the general Leibniz formula*

$$\begin{aligned}\frac{d^m}{d\theta^m}&\left(f(\theta)\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\right)\Big|_{\theta=0} \\ &= \sum_{l=0}^{\lfloor m/2 \rfloor} \binom{m}{2l} f^{(m-2l)}(0)\mathscr{R}_l(\lambda_k^{(\alpha,\beta)}).\end{aligned} \tag{3.17}$$

Proof. We can write the derivative of the Jacobi polynomials as

$$\sum_{l=0}^{\lfloor m/2 \rfloor} \binom{m}{2l} f^{(m-2l)}(\theta)\Big|_{\theta=0} \frac{d^{2l}}{d\theta^{2l}} \mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\Big|_{\theta=0}. \tag{3.18}$$

This can be written as

$$\left\{\sum_{p=0}^{m} \binom{m}{p} f^{(m-p)}(\theta)\left(\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\right)^{(p)}\right\}\Big|_{\theta=0} \tag{3.19}$$

where $\mathscr{R}_l$ are the Jacobi polynomials that are only non-zero when $p = 2l, \ l \in \mathbb{N}$. Substituting different values of m gives the result. □

We have the following expansions for $1 \le m \le 10$ odd:

- For $m = 1$ we have $f'(0)$.
- For $m = 3$ we have

$$f'''(0) + 3f'(0)\mathscr{R}_1(\lambda_k). \tag{3.20}$$

For $m = 5$ we have

$$f^{(5)}(0) + 10f'''(0)\mathscr{R}_1(\lambda_k) + 5f'(0)\mathscr{R}_2(\lambda_k). \tag{3.21}$$

For $m = 7$ we have

$$f^{(7)}(0) + 21f^{(5)}(0)\mathscr{R}_1(\lambda_k) + 35f^{(3)}(0)\mathscr{R}_2(\lambda_k) + 7f'(0)\mathscr{R}_3(\lambda_k). \tag{3.22}$$

For $m = 9$ we have

$$\begin{aligned} &f^{(9)}(0) + 36f^{(7)}(0)\mathscr{R}_1(\lambda_k) + 126f^{(5)}(0)\mathscr{R}_2(\lambda_k) + 84f^{(3)}(0)\mathscr{R}_3(\lambda_k) \\ &+ 9f'(0)\mathscr{R}_4(\lambda_k). \end{aligned} \tag{3.23}$$

These expansions can also be written as matrix inner products. For $m = 3$ we can write

$$\mathscr{Q}_3 = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & \mathsf{c}_1^1 \end{pmatrix} \begin{pmatrix} f'''(0) \\ 3f'(0) \end{pmatrix}, \begin{pmatrix} 1 \\ X \end{pmatrix} \right\rangle. \tag{3.24}$$

For $m = 5$ we can write

$$\mathscr{Q}_5 = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathsf{c}_1^1 & \mathsf{c}_1^2 \\ 0 & 0 & \mathsf{c}_2^2 \end{pmatrix} \begin{pmatrix} f^{(5)}(0) \\ 10f'''(0) \\ 5f'(0) \end{pmatrix}, \begin{pmatrix} 1 \\ X \\ X^2 \end{pmatrix} \right\rangle. \tag{3.25}$$

For $m = 7$ we can write

$$\mathscr{Q}_7 = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathsf{c}_1^1 & \mathsf{c}_1^2 & \mathsf{c}_1^3 \\ 0 & 0 & \mathsf{c}_2^2 & \mathsf{c}_2^3 \\ 0 & 0 & 0 & \mathsf{c}_3^3 \end{pmatrix} \begin{pmatrix} f^{(7)}(0) \\ 21f^{(5)}(0) \\ 35f'''(0) \\ 7f'(0) \end{pmatrix}, \begin{pmatrix} 1 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix} \right\rangle. \tag{3.26}$$

For $m = 9$ we can write

$$\mathscr{Q}_9 = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \mathsf{c}_1^1 & \mathsf{c}_1^2 & \mathsf{c}_1^3 & \mathsf{c}_1^4 \\ 0 & 0 & \mathsf{c}_2^2 & \mathsf{c}_2^3 & \mathsf{c}_2^4 \\ 0 & 0 & 0 & \mathsf{c}_3^3 & \mathsf{c}_3^4 \\ 0 & 0 & 0 & 0 & \mathsf{c}_4^4 \end{pmatrix} \begin{pmatrix} f^{(9)}(0) \\ 36f^{(7)}(0) \\ 126f^{(5)}(0) \\ 84f'''(0) \\ 9f'(0) \end{pmatrix}, \begin{pmatrix} 1 \\ X^1 \\ X^2 \\ X^3 \\ X^4 \end{pmatrix} \right\rangle. \tag{3.27}$$

The even cases can be generalised as follows:

$$\mathscr{Q}_m = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathsf{c}_1^1 & \mathsf{c}_1^2 & \mathsf{c}_1^3 & \cdots & \mathsf{c}_1^{d-1} & \mathsf{c}_1^d \\ 0 & 0 & \mathsf{c}_2^2 & \mathsf{c}_2^3 & \cdots & \mathsf{c}_2^{d-1} & \mathsf{c}_2^d \\ 0 & 0 & 0 & \mathsf{c}_3^3 & \cdots & \mathsf{c}_3^{d-1} & \mathsf{c}_3^d \\ \vdots & \vdots & \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & \vdots & \mathsf{c}_{d-1}^{d-1} & \mathsf{c}_{d-1}^d \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mathsf{c}_d^d \end{pmatrix} \begin{pmatrix} f^m(0) \\ \binom{m}{2} f^{(m-2)}(0) \\ \binom{m}{4} f^{(m-4)}(0) \\ \binom{m}{6} f^{(m-6)}(0) \\ \vdots \\ \binom{m}{m-2} f'(0) \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ X \\ X^2 \\ X^3 \\ \vdots \\ X^{d-1} \\ X^d \end{pmatrix} \right\rangle. \tag{3.28}$$

We have the following expansions for $1 \leq m \leq 10$ even:

- For $m = 2$ we have

$$f''(0) + f(0)\mathscr{R}_1(\lambda_k). \tag{3.29}$$

- For $m = 4$ we have

$$f^{(4)}(0) + 6f''(0)\mathscr{R}_1(\lambda_k) + f(0)\mathscr{R}_2(\lambda_k). \tag{3.30}$$

- For $m = 6$ we have

$$f^{(6)}(0) + 15f^{(4)}(0)\mathscr{R}_1(\lambda_k) + 15f''(0)\mathscr{R}_2(\lambda_k) + f(0)\mathscr{R}_3(\lambda_k). \tag{3.31}$$

- For $m = 8$ we have

$$\begin{aligned} &f^{(8)}(0) + 28f^{(6)}(0)\mathscr{R}_1(\lambda_k) + 70f^{(4)}(0)\mathscr{R}_2(\lambda_k) + 28f''(0)\mathscr{R}_3(\lambda_k) \\ &+ f(0)\mathscr{R}_4(\lambda_k). \end{aligned} \tag{3.32}$$

- For $m = 10$ we have

$$\begin{aligned} &f^{(10)}(0) + 45f^{(8)}(0)\mathscr{R}_1(\lambda_k) + 210f^{(6)}(0)\mathscr{R}_2(\lambda_k) \\ &+ 210f^{(4)}(0)\mathscr{R}_3(\lambda_k) + 45f''(0)\mathscr{R}_4(\lambda_k) + f(0)\mathscr{R}_5(\lambda_k). \end{aligned} \tag{3.33}$$

These expansions can also be written as matrix inner products. For $m = 2$ we can write

$$\mathscr{Q}_2 = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & \mathsf{c}_1^1 \end{pmatrix} \begin{pmatrix} f''(0) \\ f(0) \end{pmatrix}, \begin{pmatrix} 1 \\ X \end{pmatrix} \right\rangle. \tag{3.34}$$

For $m = 4$ we can write

$$\mathscr{Q}_4 = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathsf{c}_1^1 & \mathsf{c}_1^2 \\ 0 & 0 & \mathsf{c}_2^2 \end{pmatrix} \begin{pmatrix} f^{(4)}(0) \\ 6f^{(2)}(0) \\ f(0) \end{pmatrix}, \begin{pmatrix} 1 \\ X \\ X^2 \end{pmatrix} \right\rangle. \tag{3.35}$$

For $m = 6$ we can write

$$\mathscr{Q}_6 = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathsf{c}_1^1 & \mathsf{c}_1^2 & \mathsf{c}_1^3 \\ 0 & 0 & \mathsf{c}_2^2 & \mathsf{c}_2^3 \\ 0 & 0 & 0 & \mathsf{c}_3^3 \end{pmatrix} \begin{pmatrix} f^{(6)}(0) \\ 15f^{(4)}(0) \\ 15f''(0) \\ f(0) \end{pmatrix}, \begin{pmatrix} 1 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix} \right\rangle. \tag{3.36}$$

For $m = 8$ we can write

$$\mathscr{Q}_8 = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \mathsf{c}_1^1 & \mathsf{c}_1^2 & \mathsf{c}_1^3 & \mathsf{c}_1^4 \\ 0 & 0 & \mathsf{c}_2^2 & \mathsf{c}_2^3 & \mathsf{c}_2^4 \\ 0 & 0 & 0 & \mathsf{c}_3^3 & \mathsf{c}_3^4 \\ 0 & 0 & 0 & 0 & \mathsf{c}_4^4 \end{pmatrix} \begin{pmatrix} f^{(8)}(0) \\ 28f^{(6)}(0) \\ 70f^{(4)}(0) \\ 28f''(0) \\ f(0) \end{pmatrix}, \begin{pmatrix} 1 \\ X^1 \\ X^2 \\ X^3 \\ X^4 \end{pmatrix} \right\rangle. \tag{3.37}$$

For $m = 10$ we can write

$$\mathscr{Q}_{10} = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathsf{c}_1^1 & \mathsf{c}_1^2 & \mathsf{c}_1^3 & \mathsf{c}_1^4 & \mathsf{c}_1^5 \\ 0 & 0 & \mathsf{c}_2^2 & \mathsf{c}_2^3 & \mathsf{c}_2^4 & \mathsf{c}_2^5 \\ 0 & 0 & 0 & \mathsf{c}_3^3 & \mathsf{c}_3^4 & \mathsf{c}_3^5 \\ 0 & 0 & 0 & 0 & \mathsf{c}_4^4 & \mathsf{c}_4^5 \\ 0 & 0 & 0 & 0 & 0 & \mathsf{c}_5^5 \end{pmatrix} \begin{pmatrix} f^{(10)}(0) \\ 45f^{(8)}(0) \\ 210f^{(6)}(0) \\ 210f^{(4)}(0) \\ 45f^{''(0)} \\ f(0) \end{pmatrix}, \begin{pmatrix} 1 \\ X^1 \\ X^2 \\ X^3 \\ X^4 \\ X^5 \end{pmatrix} \right\rangle. \tag{3.38}$$

The even cases can be generalised as follows:

$$\mathscr{Q}_m = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathsf{c}_1^1 & \mathsf{c}_1^2 & \mathsf{c}_1^3 & \cdots & \mathsf{c}_1^{d-1} & \mathsf{c}_1^d \\ 0 & 0 & \mathsf{c}_2^2 & \mathsf{c}_2^3 & \cdots & \mathsf{c}_2^{d-1} & \mathsf{c}_2^d \\ 0 & 0 & 0 & \mathsf{c}_3^3 & \cdots & \mathsf{c}_3^{d-1} & \mathsf{c}_3^d \\ \vdots & \vdots & \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & \vdots & \mathsf{c}_{d-1}^{d-1} & \mathsf{c}_{d-1}^d \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mathsf{c}_d^d \end{pmatrix} \begin{pmatrix} f^m(0) \\ \binom{m}{2} f^{(m-2)}(0) \\ \binom{m}{4} f^{(m-4)}(0) \\ \binom{m}{6} f^{(m-6)}(0) \\ \vdots \\ \binom{m}{m-2} f''(0) \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ X \\ X^2 \\ X^3 \\ \vdots \\ X^{d-1} \\ X^d \end{pmatrix} \right\rangle. \tag{3.39}$$

Let $\alpha, \beta > -1$ and $k_1, k_2, \ldots k_m \geq 0$ be fixed integers. Consider the differential action

$$\frac{d^{2l}}{2\theta^{2l}} \mathsf{P}^{(\alpha,\beta)}_{k_1,k_2,\ldots,k_m}(\cos\theta)\Big|_{\theta=0} := \frac{d^{2l}}{d\theta^{2l}} \Big[\mathscr{P}^{(\alpha,\beta)}_{k_1}(\cos\theta) \mathscr{P}^{(\alpha,\beta)}_{k_2}(\cos\theta) \ldots \mathscr{P}^{(\alpha,\beta)}_{k_m}(\cos\theta) \Big]. \tag{3.40}$$

Proposition 3.6.2. *(Spectral polynomials in two variables $\lambda_{k_1}, \lambda_{k_2}$)*

Consider the product of Jacobi polynomial $\mathsf{P}^{(\alpha,\beta)}_{k_1,k_2}(\cos\theta) = \mathscr{P}^{(\alpha,\beta)}_{k_1}(\cos\theta)\mathscr{P}^{(\alpha,\beta)}_{k_2}(\cos\theta)$ with $k_1, k_2 \geq 0, \alpha, \beta > -1$. Then for any integer $l \geq 1$ we have

$$\begin{aligned} \frac{d^{2l}}{2\theta^{2l}} \mathsf{P}^{(\alpha,\beta)}_{k_1,k_2}(\cos\theta)\Big|_{\theta=0} &= \sum_{p=0}^{l} \binom{2l}{2p} \sum_{i=0}^{l-p} \sum_{j=0}^{p} c_i^{l-p}(\alpha,\beta) c_j^p(\alpha,\beta) [\lambda^{(\alpha,\beta)}_{k_1}]^i [\lambda^{(\alpha,\beta)}_{k_2}]^j \\ &= \mathscr{R}^{(\alpha,\beta)}_l(\lambda_{k_1}, \lambda_{k_2}). \end{aligned} \tag{3.41}$$

The scalars $(c_j^l(\alpha,\beta)\ :\ 1 \le j \le l)$ *are the usual Jacobi coefficients,* $\lambda_{k_j} = (k_j(k_j+\alpha+\beta+1)\ :\ k_j \ge 0)$, $j=1,2$, *are the eigenvalues of the Jacobi operator and* $\mathscr{R}_l^{(\alpha,\beta)} = \mathscr{R}_l^{(\alpha,\beta)}(X_1,X_2)$ *are* l*-degree polynomials in* X_1 *and* X_2*.*

Proof. The spectral derivative (3.40) with $m=2$ reduces to the differential action

$$\frac{d^{2l}}{d\theta^{2l}}\mathsf{P}_{k_1,k_2}^{(\alpha,\beta)}(\cos\theta)\Big|_{\theta=0} := \frac{d^{2l}}{d\theta^{2l}}\left[\mathscr{P}_{k_1}^{(\alpha,\beta)}(\cos\theta)\mathscr{P}_{k_2}^{(\alpha,\beta)}(\cos\theta)\right]\Big|_{\theta=0}. \tag{3.42}$$

For simplicity of notation we shall let $y_j = \mathscr{P}_{k_j}^{(\alpha,\beta)}$ for $j=1,2, l\ge 1$. Then

$$\begin{aligned}\mathscr{R}_l &= \frac{d^{2l}}{d\theta^{2l}}[y_1(\cos\theta)y_2(\cos\theta)]\Big|_{\theta=0}\\ &= \sum_{r=0}^{2l}\binom{2l}{r}\frac{d^{2l-r}}{d\theta^{2l-r}}y_1(\cos\theta)\frac{d^r}{d\theta^r}y_2(\cos\theta)\Big|_{\theta=0}\\ &= \sum_{p=0}^{l}\binom{2l}{2p}\frac{d^{2l-2p}}{d\theta^{2l-2p}}y_1(\cos\theta)\frac{d^{2p}}{d\theta^{2p}}y_2(\cos\theta)\Big|_{\theta=0}.\end{aligned} \tag{3.43}$$

By proposition 3.6.1 we have

$$\mathscr{R}_l(\mu_1,\mu_2) = \sum_{p=0}^{l}\binom{2l}{2p}\sum_{i=0}^{l-p}\sum_{j=0}^{p}c_i^{l-p}c_j^p[\mu_1]^i[\mu_2]^j, \tag{3.44}$$

where we have let $\mu_j = \lambda_{k_j}^{(\alpha,\beta)}$, for $j=1,2$. Note that $c_0^m = 1$ for $m=0$ and $c_0^m = 0$ for $m \ge 1$, i.e., $c_0^m = \delta_{0m}$. $\square$

We are now interested in the expansion of the identity

$$\mathscr{R}_l(X_1,X_2) := \frac{d^{2l}}{d\theta^{2l}}\left[\mathscr{P}_{k_1}^{(\alpha,\beta)}(\cos\theta)\mathscr{P}_{k_2}^{(\alpha,\beta)}(\cos\theta)\right]\Big|_{\theta=0} \tag{3.45}$$

for $1 \leq l \leq 5$. We have the following polynomials:

$$\mathscr{R}_1(X_1, X_2) = c_1^1(X_1 + X_2), \tag{3.46}$$

$$\mathscr{R}_2(X_1, X_2) = c_1^2(X_1 + X_2) + c_2^2(X_1^2 + X_2^2) + 6(c_1^1)^2 X_1 X_2, \tag{3.47}$$

$$\begin{aligned} \mathscr{R}_3(X_1, X_2) = {} & c_1^3(X_1 + X_2) + c_2^3(X_1^2 + X_2^2) + 30c_1^2 c_1^1 X_1 X_2 \\ & + c_3^3(X_1^3 + X_2^3) + 15c_1^1 c_2^2 X_1^2 X_2 + 15c_1^1 c_2^2 X_1 X_2^2, \end{aligned} \tag{3.48}$$

$$\begin{aligned} \mathscr{R}_4(X_1, X_2) = {} & c_1^4(X_1 + X_2) + c_2^4(X_1^2 + X_2^2) + 70(c_1^2)^2 X_1 X_2 \\ & + 56c_1^1 c_1^3 X_1 X_2 + c_3^4(X_1^3 + X_2^3) + 70c_1^2 c_2^2 X_1 X_2^2 \\ & + 70c_1^2 c_2^2 X_1^2 X_2 + 28c_1^1 c_2^3(X_1^2 X_2 + X_1 X_2^2) \\ & + c_4^4(X_1^4 + X_2^4) + 70(c_2^2)^2 X_1^2 X_2^2 \\ & + 28c_1^1 c_3^3(X_1^3 X_2 + X_1 X_2^3), \end{aligned} \tag{3.49}$$

$$\begin{aligned} \mathscr{R}_5(X_1, X_2) = {} & c_1^5(X_1 + X_2) + c_2^5(X_1^2 + X_2^2) + 420c_1^2 c_1^3 X_1 X_2 \\ & + 90c_1^1 c_1^4 X_1 X_2 + c_3^5(X_1^3 + X_2^3) + 210c_1^2 c_2^3(X_1^2 X_2 + X_1 X_2^2) \\ & + 210c_1^3 c_2^2(X_1 X_2^2 + X_1^2 X_2) + 45c_1^1 c_2^4(X_1^2 X_2 + X_1 X_2^2) \\ & + c_4^5(X_1^4 + X_2^4) + 210c_1^2 c_3^3(X_1^3 X_2 + X_1 X_2^3) \\ & + 45c_1^1 c_3^4(X_1^3 X_2 + X_1 X_2^3) + 420c_2^2 c_2^3 X_1^2 X_2^2 \\ & + c_5^5(X_1^5 + X_2^5) + 210c_2^2 c_3^3(X_1^3 X_2^2 + X_1^2 X_2^3) \\ & + 45c_1^1 c_4^4(X_1^4 X_2 + X_1 X_2^4). \end{aligned} \tag{3.50}$$

These expansions can be summarised as matrix inner products.

For $l = 1$ we can write

$$\mathscr{R}_1 = \left\langle \begin{pmatrix} 0 & \mathsf{c}_1^1 \\ \mathsf{c}_1^1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ X_2 \end{pmatrix}, \begin{pmatrix} 1 \\ X_1 \end{pmatrix} \right\rangle. \tag{3.51}$$

The eigenvalues are given by $\mathsf{c}_1^1, -\mathsf{c}_1^1$. The determinant is given by $-(\mathsf{c}_1^1)^2$. The trace is 0.

For $l = 2$ we can write

$$\mathscr{R}_2 = \left\langle \begin{pmatrix} 0 & \mathsf{c}_1^2 & \mathsf{c}_2^2 \\ \mathsf{c}_1^2 & 6(\mathsf{c}_1^1)^2 & 0 \\ \mathsf{c}_2^2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ X_2 \\ X_2^2 \end{pmatrix}, \begin{pmatrix} 1 \\ X_1 \\ X_1^2 \end{pmatrix} \right\rangle. \tag{3.52}$$

The eigenvalues are given by $6(\mathsf{c}_1^1)^2, \mathsf{c}_2^2$. The determinant is given by $-6(\mathsf{c}_1^1)^2(\mathsf{c}_2^2)^2$. The trace is given by $6(\mathsf{c}_1^1)^2 + 2\mathsf{c}_2^2$.

For $l = 3$ we can write

$$\mathscr{R}_3 = \left\langle \begin{pmatrix} 0 & \mathsf{c}_1^3 & \mathsf{c}_2^3 & \mathsf{c}_3^3 \\ \mathsf{c}_1^3 & 30\mathsf{c}_1^2\mathsf{c}_1^1 & 15\mathsf{c}_1^1\mathsf{c}_2^2 & 0 \\ \mathsf{c}_2^3 & 15\mathsf{c}_1^1\mathsf{c}_2^2 & 0 & 0 \\ \mathsf{c}_3^3 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ X_2 \\ X_2^2 \\ X_2^3 \end{pmatrix}, \begin{pmatrix} 1 \\ X_1 \\ X_1^2 \\ X_1^3 \end{pmatrix} \right\rangle. \tag{3.53}$$

The eigenvalues are given by $15\mathsf{c}_1^1\mathsf{c}_2^2, \mathsf{c}_3^3$. The determinant is given by $15^2(\mathsf{c}_1^1)^2(\mathsf{c}_2^2)^2(\mathsf{c}_3^3)^2$. The trace is given by $30\mathsf{c}_1^1\mathsf{c}_2^2 + 2\mathsf{c}_3^3$.

For $l = 4$ we can write

$$\mathscr{R}_4 = \left\langle \begin{pmatrix} 0 & \mathsf{c}_1^4 & \mathsf{c}_2^4 & \mathsf{c}_3^4 & \mathsf{c}_4^4 \\ \mathsf{c}_1^4 & 70(\mathsf{c}_1^2)^2 + 56\mathsf{c}_1^1\mathsf{c}_1^3 & 70\mathsf{c}_1^2\mathsf{c}_2^2 + 28\mathsf{c}_1^1\mathsf{c}_2^3 & 28\mathsf{c}_1^1\mathsf{c}_3^3 & 0 \\ \mathsf{c}_2^4 & 70\mathsf{c}_1^2\mathsf{c}_2^2 + 28\mathsf{c}_1^1\mathsf{c}_2^3 & 70(\mathsf{c}_2^2)^2 & 0 & 0 \\ \mathsf{c}_3^4 & 28\mathsf{c}_1^1\mathsf{c}_3^3 & 0 & 0 & 0 \\ \mathsf{c}_4^4 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ X_2 \\ X_2^2 \\ X_2^3 \\ X_2^4 \end{pmatrix}, \begin{pmatrix} 1 \\ X_1 \\ X_1^2 \\ X_1^3 \\ X_1^4 \end{pmatrix} \right\rangle. \tag{3.54}$$

The eigenvalues are given by $70(\mathsf{c}_2^2)^2, 56\mathsf{c}_1^1\mathsf{c}_3^3, \mathsf{c}_4^4$. The determinant is given by $70(28)^2(\mathsf{c}_1^1)^2(\mathsf{c}_2^2)^2(\mathsf{c}_3^3)^2(\mathsf{c}_4^4)^2$. The trace is given by $70(\mathsf{c}_2^2)^2 + 56\mathsf{c}_1^1\mathsf{c}_3^3 + 2\mathsf{c}_4^4$.

For $l = 5$ we can write

$$\mathscr{R}_5 = \left\langle \begin{pmatrix} 0 & \mathsf{c}_1^5 & \mathsf{c}_2^5 & \mathsf{c}_3^5 & \mathsf{c}_4^5 & \mathsf{c}_5^5 \\ \mathsf{c}_1^5 & 420\mathsf{c}_1^2\mathsf{c}_1^3 + 90\mathsf{c}_1^1\mathsf{c}_1^4 & \begin{matrix} 210\mathsf{c}_1^2\mathsf{c}_2^3 + 210\mathsf{c}_1^3\mathsf{c}_2^2 \\ +45\mathsf{c}_1^1\mathsf{c}_2^4 \end{matrix} & \begin{matrix} 210\mathsf{c}_1^2\mathsf{c}_3^3 \\ +45\mathsf{c}_1^1\mathsf{c}_3^4 \end{matrix} & 45\mathsf{c}_1^1\mathsf{c}_4^4 & 0 \\ \mathsf{c}_2^5 & \begin{matrix} 210\mathsf{c}_1^2\mathsf{c}_2^3 + 210\mathsf{c}_1^3\mathsf{c}_2^2 \\ +45\mathsf{c}_1^1\mathsf{c}_2^4 \end{matrix} & 420\mathsf{c}_2^2\mathsf{c}_2^3 & 210\mathsf{c}_2^2\mathsf{c}_3^3 & 0 & 0 \\ \mathsf{c}_3^5 & 210\mathsf{c}_1^2\mathsf{c}_3^3 + 45\mathsf{c}_1^1\mathsf{c}_3^4 & 210\mathsf{c}_2^2\mathsf{c}_3^3 & 0 & 0 & 0 \\ \mathsf{c}_4^5 & 45\mathsf{c}_1^1\mathsf{c}_4^4 & 0 & 0 & 0 & 0 \\ \mathsf{c}_5^5 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ X_2 \\ X_2^2 \\ X_2^3 \\ X_2^4 \\ X_2^5 \end{pmatrix}, \begin{pmatrix} 1 \\ X_1 \\ X_1^2 \\ X_1^3 \\ X_1^4 \\ X_1^5 \end{pmatrix} \right\rangle. \tag{3.55}$$

The eigenvalues are given by $210\mathsf{c}_2^2\mathsf{c}_3^3, 45\mathsf{c}_1^1\mathsf{c}_4^4, \mathsf{c}_5^5$. The determinant is given by $-(45)^2(210)^2(\mathsf{c}_1^1)^2(\mathsf{c}_2^2)^2(\mathsf{c}_3^3)^2(\mathsf{c}_4^4)^2(\mathsf{c}_5^5)^2$. The trace is given by $420\mathsf{c}_2^2\mathsf{c}_3^3 + 90\mathsf{c}_1^1\mathsf{c}_4^4 + 2\mathsf{c}_5^5$. We can obtain an alternative for proposition 3.6.1 when summing Fourier expansions over $j \in \mathbb{Z}$.

Proposition 3.6.3. *Let $j \in \mathbb{Z}$. Then the result from proposition* 3.6.1 *can be defined for the Fourier expansion*

$$f(\theta) = \sum_{j \in \mathbb{Z}} \hat{f}(j) e^{ij\theta} \tag{3.56}$$

as

$$\frac{d^m}{d\theta^m}\left(f(\theta)\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\right)\Big|_{\theta=0}$$
$$= \sum_{l=0}^{\lfloor m/2\rfloor}\binom{m}{2l}\sum_{j\in\mathbb{Z}}\hat{f}(j)(ij)^{m-2l}\mathscr{R}_l(\lambda_k^{(\alpha,\beta)}). \tag{3.57}$$

We have the derivative expansion

$$\begin{aligned}\frac{d^m}{d\theta^m}\left(f(\theta)\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\right)\Big|_{\theta=0} &= \sum_{l=0}^{\lfloor m/2\rfloor}\binom{m}{2l}\sum_{j\in\mathbb{Z}}\hat{f}(j)(ij)^{m-2l}\mathscr{R}_l(\lambda_k^{(\alpha,\beta)})\\ &= \sum_{j\in\mathbb{Z}}\mathsf{q}_j^m(\lambda_k^{(\alpha,\beta)})\hat{f}(j)\end{aligned}$$

where by virtue of $\mathscr{R}_0(X)=1$ and $\mathscr{R}_l(X)=\sum_{p=1}^{l}\mathsf{c}_p^lX^p$ for $l\geq 1$ we have

$$\mathsf{q}_j^m(X) = (ij)^m + \sum_{l=1}^{\lfloor m/2\rfloor}\sum_{p=1}^{l}\binom{m}{2l}(ij)^{m-2l}\mathsf{c}_p^lX^p.$$

For $m=2$ we can write

$$\mathsf{q}_j^2(X) = -j^2 + (\mathsf{c}_1^1X^1). \tag{3.58}$$

For $m=4$ we can write

$$\mathsf{q}_j^4(X) = j^4 - 6j^2(\mathsf{c}_1^1X^1) + (\mathsf{c}_1^2X^1+\mathsf{c}_2^2X^2). \tag{3.59}$$

For $m=6$ we can write

$$\begin{aligned}\mathsf{q}_j^6(X) &= -j^6 + 15j^4(\mathsf{c}_1^1X^1) - 15j^2(\mathsf{c}_1^2X^1+\mathsf{c}_2^2X^2)\\ &\quad + (\mathsf{c}_1^3X^1+\mathsf{c}_2^3X^2+\mathsf{c}_3^3X^3).\end{aligned} \tag{3.60}$$

For $m=8$ we can write

$$\begin{aligned}\mathsf{q}_j^8(X) &= j^8 - 28j^6(\mathsf{c}_1^1X^1) + 70j^4(\mathsf{c}_1^2X^1+\mathsf{c}_2^2X^2)\\ &\quad - 28j^2(\mathsf{c}_1^3X^1+\mathsf{c}_2^3X^2+\mathsf{c}_3^3X^3) + (\mathsf{c}_1^4X^1+\mathsf{c}_2^4X^2+\mathsf{c}_3^4X^3+\mathsf{c}_4^4X^4).\end{aligned} \tag{3.61}$$

For $m=10$ we can write

$$\begin{aligned}\mathsf{q}_j^{10}(X) &= -j^{10} + 45j^8(\mathsf{c}_1^1X^1) - 210j^6(\mathsf{c}_1^2X^1+\mathsf{c}_2^2X^2)\\ &\quad + 210j^4(\mathsf{c}_1^3X^1+\mathsf{c}_2^3X^2+\mathsf{c}_3^3X^3)\\ &\quad - 45j^2(\mathsf{c}_1^4X^1+\mathsf{c}_2^4X^2+\mathsf{c}_3^4X^3+\mathsf{c}_4^4X^4)\\ &\quad + (\mathsf{c}_1^5X^1+\mathsf{c}_2^5X^2+\mathsf{c}_3^5X^3+\mathsf{c}_4^5X^4+\mathsf{c}_5^5X^5)\end{aligned} \tag{3.62}$$

This can be summarised as an inner product. Let $\mathsf{J} = ij$.

Then for $m = 2$ we can write

$$\mathsf{q}_j^2(X) = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & \mathsf{c}_1^1 \end{pmatrix} \begin{pmatrix} \mathsf{J}^2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ X^1 \end{pmatrix} \right\rangle . \tag{3.63}$$

For $m = 4$ we can write

$$\mathsf{q}_j^4(X) = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathsf{c}_1^1 & \mathsf{c}_1^2 \\ 0 & 0 & \mathsf{c}_2^2 \end{pmatrix} \begin{pmatrix} \mathsf{J}^4 \\ 6\mathsf{J}^2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ X^1 \\ X^2 \end{pmatrix} \right\rangle . \tag{3.64}$$

For $m = 6$ we can write

$$\mathsf{q}_j^6(X) = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathsf{c}_1^1 & \mathsf{c}_1^2 & \mathsf{c}_1^3 \\ 0 & 0 & \mathsf{c}_2^2 & \mathsf{c}_2^3 \\ 0 & 0 & 0 & \mathsf{c}_3^3 \end{pmatrix} \begin{pmatrix} \mathsf{J}^6 \\ 15\mathsf{J}^4 \\ 15\mathsf{J}^2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix} \right\rangle . \tag{3.65}$$

For $m = 8$ we can write

$$\mathsf{q}_j^8(X) = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \mathsf{c}_1^1 & \mathsf{c}_1^2 & \mathsf{c}_1^3 & \mathsf{c}_1^4 \\ 0 & 0 & \mathsf{c}_2^2 & \mathsf{c}_2^3 & \mathsf{c}_2^4 \\ 0 & 0 & 0 & \mathsf{c}_3^3 & \mathsf{c}_3^4 \\ 0 & 0 & 0 & 0 & \mathsf{c}_4^4 \end{pmatrix} \begin{pmatrix} \mathsf{J}^8 \\ 28\mathsf{J}^6 \\ 70\mathsf{J}^4 \\ 28\mathsf{J}^2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ X^1 \\ X^2 \\ X^3 \\ X^4 \end{pmatrix} \right\rangle . \tag{3.66}$$

For $m = 10$ we can write

$$\mathsf{q}_j^{10}(X) = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathsf{c}_1^1 & \mathsf{c}_1^2 & \mathsf{c}_1^3 & \mathsf{c}_1^4 & \mathsf{c}_1^5 \\ 0 & 0 & \mathsf{c}_2^2 & \mathsf{c}_2^3 & \mathsf{c}_2^4 & \mathsf{c}_2^5 \\ 0 & 0 & 0 & \mathsf{c}_3^3 & \mathsf{c}_3^4 & \mathsf{c}_3^5 \\ 0 & 0 & 0 & 0 & \mathsf{c}_4^4 & \mathsf{c}_4^5 \\ 0 & 0 & 0 & 0 & 0 & \mathsf{c}_5^5 \end{pmatrix} \begin{pmatrix} \mathsf{J}^{10} \\ 45\mathsf{J}^8 \\ 210\mathsf{J}^6 \\ 210\mathsf{J}^4 \\ 45\mathsf{J}^2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ X^1 \\ X^2 \\ X^3 \\ X^4 \\ X^5 \end{pmatrix} \right\rangle . \tag{3.67}$$

Let $d = \lfloor m/2 \rfloor$. For even m, we can write the generalisation in the form $\mathsf{q}_j^m(X) =$

$\langle \mathsf{Q}_d \mathscr{J}_e, X\rangle$ where

$$\mathsf{q}_j^m(X) = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathsf{c}_1^1 & \mathsf{c}_1^2 & \mathsf{c}_1^3 & \cdots & \mathsf{c}_1^{d-1} & \mathsf{c}_1^d \\ 0 & 0 & \mathsf{c}_2^2 & \mathsf{c}_2^3 & \cdots & \mathsf{c}_2^{d-1} & \mathsf{c}_2^d \\ 0 & 0 & 0 & \mathsf{c}_3^3 & \cdots & \mathsf{c}_3^{d-1} & \mathsf{c}_3^d \\ \vdots & \vdots & \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & \vdots & \mathsf{c}_{d-1}^{d-1} & \mathsf{c}_{d-1}^d \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mathsf{c}_d^d \end{pmatrix} \begin{pmatrix} \mathsf{J}^m \\ \binom{m}{2}\mathsf{J}^{m-2} \\ \binom{m}{4}\mathsf{J}^{m-4} \\ \binom{m}{6}\mathsf{J}^{m-6} \\ \vdots \\ \binom{m}{m-2}\mathsf{J}^2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ X \\ X^2 \\ X^3 \\ \vdots \\ X^{d-1} \\ X^d \end{pmatrix} \right\rangle. \tag{3.68}$$

For $m = 3$ we have

$$\mathsf{q}_j^3 = i^3 j^3 + 3ij\mathsf{c}_1^1 X^1. \tag{3.69}$$

For $m = 5$ we have

$$\mathsf{q}_j^5 = i^5 j^5 + 10 i^3 j^3 \mathsf{c}_1^1 X^1 + 5ij(\mathsf{c}_1^2 X^1 + \mathsf{c}_2^2 X^2). \tag{3.70}$$

For $m = 7$ we have

$$\begin{aligned} \mathsf{q}_j^7 &= i^7 j^7 + 21 i^5 j^5 \mathsf{c}_1^1 X^1 + 35 i^3 j^3 (\mathsf{c}_1^2 X^1 + \mathsf{c}_2^2 X^2) \\ &+ 7ij(\mathsf{c}_1^3 X^1 + \mathsf{c}_2^3 X^2 + \mathsf{c}_3^3 X^3). \end{aligned} \tag{3.71}$$

For $m = 9$ we have

$$\begin{aligned} \mathsf{q}_j^9 &= i^9 j^9 + 36 i^7 j^7 \mathsf{c}_1^1 X^1 + 126 i^5 j^5 (\mathsf{c}_1^2 X^1 + \mathsf{c}_2^2 X^2) \\ &+ 84 i^3 j^3 (\mathsf{c}_1^3 X^1 + \mathsf{c}_2^3 X^2 + \mathsf{c}_3^3 X^3) \\ &+ 9ij(\mathsf{c}_1^4 X^1 + \mathsf{c}_2^4 X^2 + \mathsf{c}_3^4 X^3 + \mathsf{c}_4^4 X^4). \end{aligned} \tag{3.72}$$

Now let $\mathsf{J} = ij$. This these polynomials can be generalised as follows. For $m = 3$ we have

$$\mathsf{q}_j^3(X) = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & \mathsf{c}_1^1 \end{pmatrix} \begin{pmatrix} \mathsf{J}^3 \\ 3\mathsf{J} \end{pmatrix}, \begin{pmatrix} 1 \\ X^1 \end{pmatrix} \right\rangle. \tag{3.73}$$

For $m = 5$ we have

$$\mathsf{q}_j^5(X) = \left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathsf{c}_1^1 & \mathsf{c}_1^2 \\ 0 & 0 & \mathsf{c}_2^2 \end{pmatrix} \begin{pmatrix} \mathsf{J}^5 \\ 10\mathsf{J}^3 \\ 5\mathsf{J} \end{pmatrix}, \begin{pmatrix} 1 \\ X^1 \\ X^2 \end{pmatrix} \right\rangle. \tag{3.74}$$

For $m = 7$ we have

$$\mathsf{q}_j^7(X) = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathsf{c}_1^1 & \mathsf{c}_1^2 & \mathsf{c}_1^3 \\ 0 & 0 & \mathsf{c}_2^2 & \mathsf{c}_2^3 \\ 0 & 0 & 0 & \mathsf{c}_3^3 \end{pmatrix} \begin{pmatrix} \mathsf{J}^7 \\ 21\mathsf{J}^5 \\ 35\mathsf{J}^3 \\ 7\mathsf{J} \end{pmatrix}, \begin{pmatrix} 1 \\ X^1 \\ X^2 \\ X^3 \end{pmatrix} \right\rangle. \tag{3.75}$$

For $m = 9$ we have

$$\mathsf{q}_j^9(X) = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \mathsf{c}_1^1 & \mathsf{c}_1^2 & \mathsf{c}_1^3 & \mathsf{c}_1^4 \\ 0 & 0 & \mathsf{c}_2^2 & \mathsf{c}_2^3 & \mathsf{c}_2^4 \\ 0 & 0 & 0 & \mathsf{c}_3^3 & \mathsf{c}_3^4 \\ 0 & 0 & 0 & 0 & \mathsf{c}_4^4 \end{pmatrix} \begin{pmatrix} \mathsf{J}^9 \\ 36\mathsf{J}^7 \\ 126\mathsf{J}^5 \\ 84\mathsf{J}^3 \\ 9\mathsf{J} \end{pmatrix}, \begin{pmatrix} 1 \\ X^1 \\ X^2 \\ X^3 \\ X^4 \end{pmatrix} \right\rangle. \tag{3.76}$$

For odd $m \geq 3$ and $d = \lfloor m/2 \rfloor$, the general matrix can be written in the form $\mathsf{q}_j^m(X) = \langle \mathsf{Q}_d \mathscr{J}_o, X \rangle$, specifically:

$$\mathsf{q}_j^m(X) = \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathsf{c}_1^1 & \mathsf{c}_1^2 & \mathsf{c}_1^3 & \cdots & \mathsf{c}_1^{d-1} & \mathsf{c}_1^d \\ 0 & 0 & \mathsf{c}_2^2 & \mathsf{c}_2^3 & \cdots & \mathsf{c}_2^{d-1} & \mathsf{c}_2^d \\ 0 & 0 & 0 & \mathsf{c}_3^3 & \cdots & \mathsf{c}_3^{d-1} & \mathsf{c}_3^d \\ \vdots & \vdots & \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \vdots & \vdots & \mathsf{c}_{d-1}^{d-1} & \mathsf{c}_{d-1}^d \\ 0 & 0 & 0 & 0 & \cdots & 0 & \mathsf{c}_d^d \end{pmatrix} \begin{pmatrix} \mathsf{J}^m \\ \binom{m}{2}\mathsf{J}^{m-2} \\ \binom{m}{4}\mathsf{J}^{m-4} \\ \binom{m}{6}\mathsf{J}^{m-6} \\ \vdots \\ \binom{m}{m-3}\mathsf{J}^3 \\ \binom{m}{m-1}\mathsf{J} \end{pmatrix}, \begin{pmatrix} 1 \\ X \\ X^2 \\ X^3 \\ \vdots \\ X^{d-1} \\ X^d \end{pmatrix} \right\rangle. \tag{3.77}$$

Here Q_d is an $(\lfloor m/2 \rfloor + 1) \times (\lfloor m/2 \rfloor + 1)$ or $(d+1) \times (d+1)$ matrix. Let $\{1, \lambda_1, \cdots, \lambda_d\}$ denote the spectrum of Q_d. Then we can see that the set of eigenvalues (the spectrum) of Q_d are given by

$$\sum(\mathsf{Q}_d) = \left\{1, \mathsf{c}_1^1, \mathsf{c}_2^2, \mathsf{c}_3^3, \cdots, \mathsf{c}_{d-1}^{d-1}, \mathsf{c}_d^d\right\}, \tag{3.78}$$

the trace is given by

$$\mathrm{tr}(\mathsf{Q}_d) = 1 + \mathsf{c}_1^1 + \mathsf{c}_2^2 + \mathsf{c}_3^3 + \cdots + \mathsf{c}_{d-1}^{d-1} + \mathsf{c}_d^d, \tag{3.79}$$

and the determinant is given by

$$\det(\mathsf{Q}_d) = 1 \times \mathsf{c}_1^1 \times \mathsf{c}_2^2 \times \mathsf{c}_3^3 \times \cdots \times \mathsf{c}_{d-1}^{d-1} \times \mathsf{c}_d^d. \tag{3.80}$$

Theorem 3.6.1. *Let Q_d denote the upper triangular matrix in* (3.77) *and let $d = \lfloor m/2 \rfloor$ for even m. Then the trace and determinant of Q_d are given respectively by*

$$\mathrm{tr}(\mathsf{Q}_d) = 1 + \sum_{l=1}^{d} (-1)^l \frac{(2l-1)!!}{2^l} \prod_{p=0}^{l-1} (c+p)^{-1} \qquad \forall l \geq 1, \tag{3.81}$$

and

$$\det(\mathsf{Q}_d) = \prod_{l=0}^{d} (-1)^l \frac{(2l-1)!!}{2^l} \prod_{p=0}^{l-1} (c+p)^{-1} \qquad \forall l \geq 1. \tag{3.82}$$

Proof. The matrix Q_l has eigenvalues (3.78), trace (3.79) and (3.80). The leading hypergeometric coefficients c^l can be written as

$$\mathsf{c}_l^l = 2^{-l}\mathsf{b}_l^l \mathsf{d}_{l,l} \prod_{p=0}^{l-1} (c+i)^{-1} = 2^{-l}\mathsf{b}_l^l \prod_{p=0}^{l-1} (c+p)^{-1} \tag{3.83}$$

where in deducing the second equality we have used $\mathsf{d}_{l,l} = 1$. We obtain the coefficients b_l^l recursively using the formula

$$\begin{aligned} \mathsf{b}_l^l &= -(l^2\mathsf{b}_l^{l-1} + (2l-1)\mathsf{b}_{l-1}^{l-1}) \\ &= (-1)^l \prod_{j=1}^{l} (2j-1) = (-1)^l (2l-1)!!, \qquad \forall l \geq 1. \end{aligned} \tag{3.84}$$

Hence by substitution it is seen that the leading hypergeometric coefficients have the explicit form

$$\mathsf{c}_l^l = (-1)^l \frac{(2l-1)!!}{2^l} \prod_{p=0}^{l-1} (c+p)^{-1} \qquad \forall l \geq 1. \tag{3.85}$$

Substituting this into (3.79) gives us

$$\operatorname{tr}(\mathsf{Q}_d) = 1 + \sum_{l=1}^{d} (-1)^l \frac{(2l-1)!!}{2^l} \prod_{p=0}^{l-1} (c+p)^{-1} \qquad \forall l \geq 1 \tag{3.86}$$

and the result for the trace follows. Similarly, substituting (3.85) into (3.80) means we can write

$$\det(\mathsf{Q}_d) = \prod_{l=1}^{d} (-1)^l \frac{(2l-1)!!}{2^l} \prod_{p=0}^{l-1} (c+p)^{-1} \qquad \forall l \geq 1 \tag{3.87}$$

giving us the result. $\square$

This can be summarised as the following corollary.

Corollary 3.6.1. *Let $\mathscr{M}$ denote any of the compact rank-one symmetric spaces $\mathbb{S}^n$, $\mathbb{R}\mathbf{P}^n$, $\mathbb{C}\mathbf{P}^n$ or $\mathbb{H}\mathbf{P}^n$. Then for $d \geq 1$ we have $\lim \operatorname{tr}(\mathsf{Q}_d) = 1$ and $\lim \det(\mathsf{Q}_d) = 0$ as $n \nearrow \infty$. For $c \geq 1$, we have $\lim \operatorname{tr}(\mathsf{Q}_d) = 1$ as $d \nearrow \infty$.*

We now have the following proposition.

Proposition 3.6.4. *Let Q_d be as in (3.77). Then for each fixed a, b, c with $c \notin \{0, -1, -2, \dots\}$ we have*

$$\lim_{d \nearrow \infty} \operatorname{tr}(\mathsf{Q}_d) = \sum_{l=0}^{\infty} \frac{(1/2)_l}{(c)_l} (-1)^l = F(1/2, 1; c; z)\Big|_{z=-1}. \tag{3.88}$$

Proof. Referring to (3.79) we have

$$\begin{aligned}
\lim_{l\nearrow\infty} \mathrm{tr}(\mathsf{Q}_d) &= 1 + \sum_{l=1}^{\infty} \mathsf{c}_l^l = 1 + \sum_{l=1}^{\infty} (-1)^l \frac{(2l-1)!!2^{-l}}{\prod_{p=0}^{l-1}(c+p)} \\
&= 1 + \sum_{l=1}^{\infty} (-1)^l \frac{\Gamma(l+1/2)\Gamma(c)}{\Gamma(c+l)\Gamma(1/2)} \\
&= \sum_{l=0}^{\infty} \frac{(1/2)_l (1)_l}{l!(c)_l} (-1)^l \qquad (3.89)
\end{aligned}$$

which is the required conclusion. ☐

Chapter 4

Spherical Twists Q as Solutions of Euler-Lagrange Equations $\mathscr{L}_U[u] = 0$, Constrained and Unconstrained

4.1 Introduction

In this chapter we will study a variational problem consisting of an energy functional

$$\mathbb{F}[u, \Omega] = \int_\Omega F(|x|, |u|^2, |\nabla u|^2)\, dx \tag{4.1}$$

and address questions on the existence, multiplicity as well as qualitative features including symmetries of its extremisers over two distinct classes of vector fields: firstly an unconstrained problem where the competing vector fields u lie in the space $\mathscr{W}^{1,p}(\Omega, \mathbb{R}^n)$ and secondly a constrained problem where the competing vector fields u lies in $\mathscr{W}^{1,p}(\Omega, \mathbb{S}^{n-1})$.

Let $F = F(r, s, \xi)$ in $\mathscr{C}^{1,2,2}([a, b] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ be a given integrand and consider the energy functional

$$\mathbb{F}[u; \Omega] = \int_\Omega F(|x|, |u|^2, |\nabla u|^2)\, dx \tag{4.2}$$

for $|x| \in [a, b]$ and $0 < a < b$. First note that the Euler-Lagrange equations in the two cases, respectively the unconstrained and constrained cases, take the forms:

$$\begin{cases} \mathscr{L}_U[u] = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{4.3}$$

where the unconstrained differential operator $\mathscr{L}_U$ is given by

$$\mathscr{L}_U[u] = \mathrm{div}[F_\xi(|x|, |u|^2, |\nabla u|^2)\nabla u] - F_s(|x|, |u|^2, |\nabla u|^2)u, \tag{4.4}$$

and likewise the constrained case is written

$$\begin{cases} \mathscr{L}_C[u] = 0 & \text{in } \Omega, \\ |u| = 1 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

with the operator $\mathscr{L}_C$ is given by

$$\mathscr{L}_C[u] = \mathrm{div}[F_\xi(|x|, 1, |\nabla u|^2)\nabla u] + F_\xi(|x|, 1, |\nabla u|^2)|\nabla u|^2 u. \tag{4.6}$$

Note that here F_s, F_ξ denote the derivatives of F with respect to the second and third variables respectively and the divergence operator on the corresponding matrix fields acts row-wise.

We will look at the particular example $F(r, s, \xi) = h(r, s)\xi^{p/2}$ with $1 < p < \infty$ and $h \in \mathscr{C}^1([a, b])$ satisfying $h > 0$ on $[a, b]$. This leads to a generalisation of the usual Dirichlet energy called the *weighted* p-energy taking the form

$$\mathbb{E}_p^h[u; \mathbb{X}^n] = \int_\Omega h(|x|, |u|^2)|\nabla u|^p \, dx \tag{4.7}$$

where the Euler-Lagrange equation in this case is as formulated by the systems (4.3) and (5.5) with the differential operators $\mathscr{L}_U$ and $\mathscr{L}_C$ respectively being given by

$$\mathscr{L}_U[u] = \frac{p}{2}\left(h\Delta_p u + |\nabla u|^{p-2}\nabla u \mathrm{div}[h]\right) - h_s u|\nabla u|^p = 0, \tag{4.8}$$

and

$$\mathscr{L}_C[u] = h\Delta_p u + |\nabla u|^{p-2}\nabla u\nabla \cdot h + h|\nabla u|^p u = 0, \tag{4.9}$$

with $\Delta_p u := \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right)$ the p-Laplacian, and h_s is the derivative in the second variable of $h = h(|x|, 1)$.

The first class of maps we examine in this chapter as solutions to the nonlinear systems (4.3) and (5.5) and are the so-called spherical twists. Recall that a *spherical twist* by definition is a map $u \in \mathscr{C}(\overline{\mathbb{X}}^n, \mathbb{S}^{n-1})$ of the form

$$u : x = r\theta \mapsto \mathbf{Q}(r)\theta = \mathbf{Q}(|x|)x|x|^{-1}, \qquad x \in \overline{\mathbb{X}}^n, \tag{4.10}$$

where $0 < a \leq r = |x| \leq b$ and $\mathbf{Q} \in \mathscr{C}([a, b], \mathbf{SO}(n))$. For obvious geometric reasons the continuous curve $[a, b] \ni r \mapsto \mathbf{Q}(r) \in \mathbf{SO}(n)$ will be referred to as the *twist path* associated with the spherical twist u. We henceforth aim to describe those twist paths $\mathbf{Q} \in \mathscr{C}^2([a,b], \mathbf{SO}(n))$ such that the corresponding spherical twist $u = \mathbf{Q}(r)x|x|^{-1}$ gives a [*classical*] solutions to the two systems of Euler-Lagrange equations. This as will be seen involves a study of the geodesics of the compact lie group $\mathbf{SO}(n)$ and their links with geodesics on the sphere.

The second class of maps we examine in this paper as solutions to the systems (4.3) and (5.5) are the spherical whirls (or whirls for simplicity). These by definition are maps $u \in \mathscr{C}(\overline{\mathbb{X}}^n, \mathbb{S}^{n-1})$ of the form

$$u : x \mapsto \mathbf{Q}(\rho_1, \dots, \rho_N)x|x|^{-1}, \quad x \in \overline{\mathbb{X}}^n, \tag{4.11}$$

where $\mathbf{Q} = \mathbf{Q}(\rho_1, \dots, \rho_N)$ is a continuous $\mathbf{SO}(n)$-valued map depending on the spatial variable $x = (x_1, \dots, x_n)$ through the 2-plane variables $\rho = (\rho_1, \dots, \rho_N)$, that, depending on the dimension n being even or odd, we have the description:

[**a**] (n even) writing $n = 2N$ we set $k = N$ and then

$$\rho_j = \sqrt{x_{2j-1}^2 + x_{2j}^2} \quad \text{for } 1 \le j \le N. \tag{4.12}$$

[**b**] (n odd) writing $n = 2N-1$ we set $k = N-1$ and then

$$\rho_j := \begin{cases} \sqrt{x_{2j-1}^2 + x_{2j}^2} & \text{for } 1 \le j \le N-1, \\ x_n & \text{for } j = N. \end{cases} \tag{4.13}$$

4.2 Spherical Twists as Extremisers of a Restricted Energy: An ODE for Twist Paths

Let $\mathbb{X}^n = \mathbb{X}^n[a,b] = \{x \in \mathbb{R}^n : a < |x| < b\}$ with $0 < a < b < \infty$ be a generalised annulus and consider the $\mathbb{F}$-energy functional as defined earlier by the integral

$$\mathbb{F}[u; \mathbb{X}^n] := \int_{\mathbb{X}^n} F(|x|, |u|^2, |\nabla u|^2)\, dx. \tag{4.14}$$

For $\mathbb{X}^n$ as above, we introduce the space of admissable twist paths

$$\mathscr{B}_R^p = \mathscr{B}_R^p([a,b]) = \big\{\mathbf{Q} \in W^{1,p}((a,b), \mathbf{SO}(n)) : \mathbf{Q}(a) = \mathbf{I}_n, \mathbf{Q}(b) = \mathbf{R}\big\}. \tag{4.15}$$

Proposition 4.2.1 (Key identities for $u(x) = f(r)\mathbf{Q}(r)v(x)$)**.** *Let $r = |x|$ and $\theta = x|x|^{-1}$. Suppose $u = f\mathbf{Q}v$ is a spherical twist defined by a twice continuously differentiable twist path $\mathbf{Q} = \mathbf{Q}(r)$, function $f \in \mathscr{C}^2([a,b], \mathbb{R})$ and $v \in \mathscr{C}^2(\mathbb{X}^n, \mathbb{S}^{n-1})$. Then the following identities hold:*

(i) $\nabla u = \dot{f}\mathbf{Q}v \otimes \theta + f\dot{\mathbf{Q}}v \otimes \theta + f\mathbf{Q}\nabla v,$

(ii) $|\nabla u|^2 = f^2\left(|\dot{\mathbf{Q}}v|^2 + |\nabla v|^2 + 2\langle \dot{\mathbf{Q}}v \otimes \theta, \nabla v\mathbf{Q}\rangle\right) + \dot{f}^2,$

(iii)
$$\Delta u = f\left(\ddot{\mathbf{Q}}v + 2\dot{\mathbf{Q}}\nabla v\theta + (n-1)r^{-1}\dot{\mathbf{Q}}v + \mathbf{Q}\Delta v\right) + \ddot{f}\mathbf{Q}v$$
$$+ \dot{f}\left(2\dot{\mathbf{Q}}v + 2\mathbf{Q}\nabla v\theta + (n-1)r^{-1}\mathbf{Q}v\right),$$

(iv)
$$\Delta_p u = |\nabla u|^{p-2}\Big[2f\dot{f}\left(|\dot{\mathbf{Q}}v|^2\theta + |\nabla v|^2\theta + \langle \dot{\mathbf{Q}}v \otimes \theta, \nabla v\mathbf{Q}\rangle\right) + 2\dot{f}\ddot{f}$$
$$+ f^2\left(\nabla|\dot{\mathbf{Q}}v|^2 + \nabla|\nabla v|^2 + \nabla\langle \dot{\mathbf{Q}}v \otimes \theta, \nabla v\mathbf{Q}\rangle\right)\Big]$$
$$+ |\nabla u|^{p-4}\frac{p-2}{2}\Big[\dot{f}\mathbf{Q}v \otimes \theta + f\dot{\mathbf{Q}}v \otimes \theta + f\mathbf{Q}\nabla v\Big]\Big[2f\dot{f}\Big(|\dot{\mathbf{Q}}v|^2\theta$$
$$+ \langle \dot{\mathbf{Q}}v \otimes \theta, \nabla v\mathbf{Q}\rangle\theta + |\nabla v|^2\theta\Big) + 2\dot{f}\ddot{f} + f^2\Big(\nabla|\dot{\mathbf{Q}}v|^2 + \nabla|\nabla v|^2$$
$$+ \nabla\langle \dot{\mathbf{Q}}v \otimes \theta, \nabla v\mathbf{Q}\rangle\Big)\Big].$$

Writing F_ξ as the derivative in the third variable of $F = F(r,s,\xi)$, and $F_{\xi r}$, $F_{\xi s}$ and $F_{\xi\xi}$ the derivatives in the first, second, and third variables respectively of F_ξ, and also noting that $|u| = |f|$, we have

(v) $\operatorname{div}\left[F_\xi(r, f^2, |\nabla u|^2)\nabla u\right]$
$$= F_\xi\Big[2f\dot{f}\left(|\dot{\mathbf{Q}}v|^2\theta + |\nabla v|^2\theta + \langle \dot{\mathbf{Q}}v \otimes \theta, \nabla v\mathbf{Q}\rangle\right) + 2\dot{f}\ddot{f}$$
$$+ f^2\left(\nabla|\dot{\mathbf{Q}}v|^2 + \nabla|\nabla v|^2 + \nabla\langle \dot{\mathbf{Q}}v \otimes \theta, \nabla v\mathbf{Q}\rangle\right)\Big]$$
$$+ F_{\xi r}\left[\dot{f}\mathbf{Q}v + f\dot{\mathbf{Q}}v + f\mathbf{Q}\nabla v\theta\right] + 2f\dot{f}F_{\xi s}\left[\dot{f}\mathbf{Q}v + f\dot{\mathbf{Q}}v + f\mathbf{Q}\nabla v\theta\right]$$
$$+ F_{\xi\xi}\Big[\dot{f}\mathbf{Q}v \otimes \theta + f\dot{\mathbf{Q}}v \otimes \theta + f\mathbf{Q}\nabla v\Big]\Big[2f\dot{f}\Big(|\dot{\mathbf{Q}}v|^2\theta + \langle \dot{\mathbf{Q}}v \otimes \theta, \nabla v\mathbf{Q}\rangle\theta$$
$$+ |\nabla v|^2\theta\Big) + 2\dot{f}\ddot{f} + f^2\left(\nabla|\dot{\mathbf{Q}}v|^2 + \nabla|\nabla v|^2 + \nabla\langle \dot{\mathbf{Q}}v \otimes \theta, \nabla v\mathbf{Q}\rangle\right)\Big].$$

Proof. Taking the gradient of $u = f\mathbf{Q}v$ results in the first identity, and then taking the divergence of this gives the third identity. For the second, we take the Hilbert-Schmidt norm, writing $|\nabla u|^2 = \operatorname{tr}([\nabla u][\nabla u]^t)$. Evaluating this for ∇u as in the first identity gives the result. The fifth identity can be evaluated by writing

$$\operatorname{div}\left[F_\xi \nabla u\right] = F_\xi \Delta u + \nabla u\left(F_{\xi r}\theta + F_{\xi s}\nabla\left[f^2\right] + F_{\xi\xi}\nabla|\nabla u|^2\right). \tag{4.16}$$

We have $\nabla f^2 = 2f\dot{f}\theta$, as well as

$$\nabla|\nabla u|^2 = 2f\dot{f}\left(|\dot{\mathbf{Q}}v|^2\theta + |\nabla v|^2\theta + \langle \dot{\mathbf{Q}}v \otimes \theta, \nabla v\mathbf{Q}\rangle\theta\right) + 2\dot{f}\ddot{f}$$
$$+ f^2\left(\nabla|\dot{\mathbf{Q}}v|^2 + \nabla|\nabla v|^2 + \nabla\langle \dot{\mathbf{Q}}v \otimes \theta, \nabla v\mathbf{Q}\rangle\right). \tag{4.17}$$

Substituting these, together with earlier identities, into (4.16) gives the result. For the fourth identity, we consider a special case of the fifth. Taking $F(r,s,\xi) = 2p^{-1}\xi^{p/2}$ in the fifth identity results in the p-Laplacian, and so the fourth identity follows. □

Using the above proposition we can write the unconstrained and constrained operators $\mathscr{L}_U$ and $\mathscr{L}_C$ respectively as

$$\begin{aligned}
&\mathscr{L}_U[u = f\mathbf{Q}v] \\
&\quad = F_\xi\Big[2f\dot{f}\left(|\dot{\mathbf{Q}}v|^2\theta + |\nabla v|^2\theta + \langle\dot{\mathbf{Q}}v\otimes\theta, \nabla v\mathbf{Q}\rangle\right) + 2\dot{f}\ddot{f} \\
&\qquad + f^2\left(\nabla|\dot{\mathbf{Q}}v|^2 + \nabla|\nabla v|^2 + \nabla\langle\dot{\mathbf{Q}}v\otimes\theta, \nabla v\mathbf{Q}\rangle\right)\Big] - fF_s\mathbf{Q}v \\
&\qquad + F_{\xi r}\left[\dot{f}\mathbf{Q}v + f\dot{\mathbf{Q}}v + f\mathbf{Q}\nabla v\theta\right] + 2f\dot{f}F_{\xi s}\left[\dot{f}\mathbf{Q}v + f\dot{\mathbf{Q}}v + f\mathbf{Q}\nabla v\theta\right] \\
&\qquad + F_{\xi\xi}\Big[\dot{f}\mathbf{Q}v\otimes\theta + f\dot{\mathbf{Q}}v\otimes\theta + f\mathbf{Q}\nabla v\Big]\Big[2f\dot{f}\Big(|\dot{\mathbf{Q}}v|^2\theta + \langle\dot{\mathbf{Q}}v\otimes\theta, \nabla v\mathbf{Q}\rangle\theta \\
&\qquad + |\nabla v|^2\theta\Big) + 2\dot{f}\ddot{f} + f^2\left(\nabla|\dot{\mathbf{Q}}v|^2 + \nabla|\nabla v|^2 + \nabla\langle\dot{\mathbf{Q}}v\otimes\theta, \nabla v\mathbf{Q}\rangle\right)\Big],
\end{aligned} \tag{4.18}$$

and in the constrained case where $f = 1$,

$$\begin{aligned}
&\mathscr{L}_C[u = \mathbf{Q}v] \\
&\quad = F_\xi\Big[\nabla|\dot{\mathbf{Q}}v|^2 + \nabla|\nabla v|^2 + \nabla\langle\dot{\mathbf{Q}}v\otimes\theta, \nabla v\mathbf{Q}\rangle\Big] \\
&\qquad + F_{\xi r}\left[\dot{\mathbf{Q}}v + \mathbf{Q}\nabla v\theta\right] + F_\xi\left[|\dot{\mathbf{Q}}v|^2 + |\nabla v|^2 + 2\langle\dot{\mathbf{Q}}v\otimes\theta, \nabla v\mathbf{Q}\rangle\right]\mathbf{Q}v \\
&\qquad + F_{\xi\xi}\Big[\dot{\mathbf{Q}}v\otimes\theta + \mathbf{Q}\nabla v\Big]\Big[\left(\nabla|\dot{\mathbf{Q}}v|^2 + \nabla|\nabla v|^2 + \nabla\langle\dot{\mathbf{Q}}v\otimes\theta, \nabla v\mathbf{Q}\rangle\right)\Big].
\end{aligned} \tag{4.19}$$

4.3 Specialised Twist Paths: The Constrained Case

In this section we specialise our function $u = u(x)$ in such a way that $|u| = 1$. Setting $f \equiv 1$, we first consider general $v \in \mathscr{C}^2(\mathbb{X}^n, \mathbb{S}^{n-1})$ and $\mathbf{Q}$ a twist path. We then take $v(x) = x|x|^{-1}$ with unchanged $\mathbf{Q}$, followed by the case where v is unchanged and $\mathbf{Q} = \exp(\mathscr{G}\mathbf{H})$ for some real valued function $\mathscr{G} = \mathscr{G}(r)$ and constant skew-symmetric matrix $\mathbf{H}$.

4.3.1 The General Case : $u = \mathbf{Q}(r)v(x)$

Proposition 4.3.1 (Key identities for $u = \mathbf{Q}(r)v(x)$)**.** *Again let $r = |x|$ and $\theta = x|x|^{-1}$ for $x \in \mathbb{X}^n$. Consider $u = \mathbf{Q}(r)v(x)$ for $v \in \mathscr{C}^2(\mathbb{X}^n, \mathbb{S}^{n-1})$. Then the following identities hold:*

(i) $\nabla u = \dot{\mathbf{Q}}v\otimes\theta + \mathbf{Q}\nabla v$,

(ii) $|\nabla u|^2 = |\dot{\mathbf{Q}}v|^2 + 2\langle\dot{\mathbf{Q}}v\otimes\theta, \mathbf{Q}\nabla v\rangle + |\nabla v|^2$,

(iii) $\Delta u = \ddot{\mathbf{Q}}v + (n-1)\dot{\mathbf{Q}}vr^{-1} + 2\dot{\mathbf{Q}}\nabla v\theta + \mathbf{Q}\Delta v$,

(iv) $\Delta_p u = |\nabla u|^{p-2}\left(\ddot{\mathbf{Q}}v + (n-1)\dot{\mathbf{Q}}vr^{-1} + 2\dot{\mathbf{Q}}\nabla v\theta + \mathbf{Q}\Delta v\right)$
$+ |\nabla u|^{p-4}\frac{p-2}{2}\left[\nabla|\dot{\mathbf{Q}}v|^2 + \nabla|\nabla v|^2 + \nabla\langle\dot{\mathbf{Q}}v\otimes\theta, \mathbf{Q}\nabla v\rangle\right]$
$\times\left(\dot{\mathbf{Q}}v\otimes\theta + \mathbf{Q}\nabla v\right),$

and writing F_ξ *for the derivative in the third variable of* $F(r,1,|\nabla u|^2)$*, as well as* $F_{\xi r}$ *and* $F_{\xi\xi}$ *the derivatives in the first and third variables of* F_ξ*, we have*

(v) $\operatorname{div}\left[F_\xi(r,1,|\nabla u|^2)\nabla u\right]$

$$= F_{\xi r}\left(\dot{\mathbf{Q}}v + \mathbf{Q}\nabla v\theta\right) + F_\xi\left(\ddot{\mathbf{Q}}v + (n-1)\dot{\mathbf{Q}}vr^{-1} + 2\dot{\mathbf{Q}}\nabla v\theta + \mathbf{Q}\Delta v\right) + F_{\xi\xi}\left[\nabla|\dot{\mathbf{Q}}v|^2 + \nabla|\nabla v|^2 + \nabla\langle\dot{\mathbf{Q}}v\otimes\theta, \mathbf{Q}\nabla v\rangle\right]\left(\dot{\mathbf{Q}}v\otimes\theta + \mathbf{Q}\nabla v\right).$$

Using Proposition 4.3.1 we can write the constrained operator $\mathscr{L}_C[u = \mathbf{Q}v]$ as

$$\begin{aligned}\mathscr{L}_C[u] = {} & F_{\xi r}\left(\dot{\mathbf{Q}}v + \mathbf{Q}\nabla v\theta\right) + F_\xi\left(\ddot{\mathbf{Q}}v + (n-1)\dot{\mathbf{Q}}vr^{-1} + 2\dot{\mathbf{Q}}\nabla v\theta + \mathbf{Q}\Delta v\right)\\ & + F_{\xi\xi}\left[\nabla|\dot{\mathbf{Q}}v|^2 + \nabla|\nabla v|^2 + \nabla\langle\dot{\mathbf{Q}}v\otimes\theta, \mathbf{Q}\nabla v\rangle\right]\left(\dot{\mathbf{Q}}v\otimes\theta + \mathbf{Q}\nabla v\right)\\ & + F_\xi\left[|\dot{\mathbf{Q}}v|^2 + 2\big\langle\dot{\mathbf{Q}}v\otimes\theta, \mathbf{Q}\nabla v\big\rangle + |\nabla v|^2\right]\mathbf{Q}v. \end{aligned}\tag{4.20}$$

With $u = \mathbf{Q}(r)v(x)$, the $\mathbb{F}$-energy in this case becomes

$$\begin{aligned}\mathbb{F}[u;\mathbb{X}^n] &= \int_{\mathbb{X}^n} F(|x|,|u|^2,|\nabla u|^2)\,dx\\ &= \int_a^b\int_{\mathbb{S}^{n-1}} F\left(r,1,|\mathbf{Q}^t\dot{\mathbf{Q}}v|^2 + 2\big\langle\mathbf{Q}^t\dot{\mathbf{Q}}v\otimes\theta,\nabla v\big\rangle + |\nabla v|^2\right)r^{n-1}\,d\mathcal{H}^{n-1}(\theta)dr\\ &= \int_a^b J(r,\mathbf{Q}^t\dot{\mathbf{Q}},v,\nabla v)r^{n-1}\,dr =: \mathbb{J}[\mathbf{Q},v;(a,b)] \end{aligned}\tag{4.21}$$

where $J = J(r,\mathbf{A},g,\mathbf{B}) \in \mathscr{C}^{1,2,2,1}([a,b]\times\mathbb{R}^{n\times n}\times\mathbb{S}^n\times\mathrm{T}\mathbb{S}^{n-1})$ is given by

$$J(r,\mathbf{A},g,\mathbf{B}) := \int_{\mathbb{S}^{n-1}} F\left(r,1,|\mathbf{A}g|^2 + 2\big\langle\mathbf{A}g\otimes\theta,\mathbf{B}\big\rangle + |\mathbf{B}|^2\right)r^{n-1}\,d\mathcal{H}^{n-1}(\theta). \tag{4.22}$$

4.3.2 The Case $v(x) = x|x|^{-1} = \theta$: $u = \mathbf{Q}\theta$

Proposition 4.3.2 (Key identities for $u = \mathbf{Q}\theta$)**.** *Let* $r = |x|$ *and* $\theta = x|x|^{-1}$*. Suppose* $u = \mathbf{Q}(r)\theta$ *is a spherical twist defined by a twice continuously differentiable twist path* $\mathbf{Q}$*. Then the following identities hold for both the constrained and unconstrained cases:*

(i) $\nabla u = r^{-1}\left(\mathbf{Q} + (r\dot{\mathbf{Q}} - \mathbf{Q})\theta\otimes\theta\right),$

(ii) $|\nabla u|^2 = \dfrac{n-1}{r^2} + |\dot{\mathbf{Q}}\theta|^2,$

(iii) $\Delta u = \left((n-1)(r\dot{\mathbf{Q}} - \mathbf{Q}) + r^2\ddot{\mathbf{Q}}\right)\theta r^{-2},$

(iv) $\Delta_p u = |\nabla u|^{p-2} \left((n-1)(r\dot{\mathbf{Q}} - \mathbf{Q}) + r^2 \ddot{\mathbf{Q}} \right) \theta r^{-2}$
$$+ |\nabla u|^{p-4} \frac{p-2}{2r} \left(\mathbf{Q} + (r\dot{\mathbf{Q}} - \mathbf{Q})\theta \otimes \theta \right) \left(\nabla |\dot{\mathbf{Q}}\theta|^2 - 2(n-1)r^{-3}\theta \right),$$

and letting F_ξ denote the derivative in the third variable of $F(r, s, \xi)$, and $F_{\xi r}$, and $F_{\xi\xi}$ denote the derivatives with respect to the third and first variables, third and second variables, and a double derivative in the third variable respectively,

(v) $\operatorname{div} \left[F_\xi(r, 1, |\nabla u|^2) \nabla u \right]$
$$= F_\xi \left[(n-1)(r\dot{\mathbf{Q}} - \mathbf{Q}) + r^2 \ddot{\mathbf{Q}} \right] \theta r^{-2} + F_{\xi r} \dot{\mathbf{Q}}\theta$$
$$+ F_{\xi\xi} \left[\nabla |\dot{\mathbf{Q}}\theta|^2 - 2(n-1)r^{-3}\theta \right] \left(\mathbf{Q} + (r\dot{\mathbf{Q}} - \mathbf{Q})\theta \otimes \theta \right) r^{-1}.$$

The constrained operator $\mathscr{L}_C$ in this case becomes

$$\begin{aligned} &\mathscr{L}_C[u = \mathbf{Q}\theta] \\ &= F_\xi \left[\left((n-1)(r\dot{\mathbf{Q}} - \mathbf{Q}) + r^2 \ddot{\mathbf{Q}} \right) \theta r^{-2} + \left((n-1)r^{-2} + |\dot{\mathbf{Q}}\theta|^2 \right) \mathbf{Q} v \right] + F_{\xi r} \dot{\mathbf{Q}}\theta \\ &\quad + F_{\xi\xi} \left[\nabla |\dot{\mathbf{Q}}\theta|^2 - 2(n-1)r^{-3}\theta \right] \left(\mathbf{Q} + (r\dot{\mathbf{Q}} - \mathbf{Q})\theta \otimes \theta \right) r^{-1}. \end{aligned} \tag{4.23}$$

By referring to the calculations in Proposition 5.2.1 we can write the $\mathbb{F}$-energy of a spherical twist $u = \mathbf{Q}(r)x|x|^{-1}$ as

$$\begin{aligned} \mathbb{F}[\mathbf{Q}(r)x|x|^{-1}; \mathbb{X}^n] &= \int_{\mathbb{X}^n} F(|x|, 1, |\nabla u|^2)\, dx \\ &= \int_a^b \int_{\mathbb{S}^{n-1}} F\left(r, 1, \frac{n-1}{r^2} + |\dot{\mathbf{Q}}\theta|^2 \right) r^{n-1}\, dr d\mathcal{H}^{n-1}(\theta) \\ &= \int_a^b E(r, \dot{\mathbf{Q}}) r^{n-1}\, dr =: \mathbb{E}[\mathbf{Q}; (a,b)] \end{aligned} \tag{4.24}$$

where the integrand $E = E(r, \mathbf{A}) \in \mathscr{C}^{1,1}([a,b] \times \mathbb{R}^{n\times n})$ is given by

$$E(r, \mathbf{A}) := \int_{\mathbb{S}^{n-1}} F\left(r, 1, \frac{n-1}{r^2} + |\mathbf{A}\theta|^2 \right) d\mathcal{H}^{n-1}(\theta). \tag{4.25}$$

Lemma 4.3.1. *The Euler-Lagrange equation associated with the energy $\mathbb{E}$ defined by* (4.24) *over the space of admissible twist paths $\mathscr{B}^p_{\mathbf{R}}$ is given by*

$$\int_{\mathbb{S}^{n-1}} \frac{d}{dr} \left\{ r^{n-1} F_\xi \left(r, 1, \frac{n-1}{r^2} + |\dot{\mathbf{Q}}\theta|^2 \right) \left[\dot{\mathbf{Q}}\theta \otimes \mathbf{Q}\theta - \mathbf{Q}\theta \otimes \dot{\mathbf{Q}}\theta \right] \right\} d\mathcal{H}^{n-1}(\theta) = 0. \tag{4.26}$$

Proof. First fix $\mathbf{Q}$ and for $\varepsilon > 0$ define the variation $\mathbf{Q}_\varepsilon = \mathbf{Q} + \varepsilon(\mathbf{F} - \mathbf{F}^t)\mathbf{Q}$ where $\mathbf{F} \in \mathscr{C}_0^\infty((a,b), \mathbb{M}^{n\times n})$ is arbitrary. Then to the first order in ε one can compute that $\mathbf{Q}_\varepsilon$

takes values in $\mathbf{SO}(n)$. Differentiating with respect to ε and then setting $\varepsilon = 0$ gives

$$
\begin{aligned}
0 = \frac{d}{d\varepsilon}\mathbb{E}(\mathbf{Q}_\varepsilon;(a,b))\Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\int_a^b E(r,\dot{\mathbf{Q}}_\varepsilon)r^{n-1}dr\Big|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\int_a^b\int_{\mathbb{S}^{n-1}} Fr^{n-1}d\mathcal{H}^{n-1}(\theta)dr\Big|_{\varepsilon=0} \\
&= \int_a^b\int_{\mathbb{S}^{n-1}} F_\xi\frac{d}{d\varepsilon}\left(|\dot{\mathbf{Q}}_\varepsilon\theta|^2\right)r^{n-1}d\mathcal{H}^{n-1}(\theta)dr\Big|_{\varepsilon=0} \\
&= \int_a^b\int_{\mathbb{S}^{n-1}} 2F_\xi\langle\dot{\mathbf{Q}}\theta,(\dot{\mathbf{F}}-\dot{\mathbf{F}}^t)\mathbf{Q}\theta\rangle r^{n-1}d\mathcal{H}^{n-1}(\theta)dr \\
&= \int_a^b\langle\int_{\mathbb{S}^{n-1}} -2\frac{d}{dr}\left\{r^{n-1}F_\xi\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta\right\}d\mathcal{H}^{n-1}(\theta),\dot{\mathbf{F}}-\dot{\mathbf{F}}^t\rangle dr,
\end{aligned}
\tag{4.27}
$$

where we have written $F = F(r,1,(n-1)r^{-2}+|\dot{\mathbf{Q}}\theta|^2)$, and F_ξ as the derivative in the third variable of this. The integrand of the outer integral must be zero, and since this is true for all $\mathbf{F}\in\mathscr{C}_0^\infty((a,b),\mathbb{M}^{n\times n})$ we have that the skew-symmetric part of the spherical integral must also be zero, and so the result is proven. □

In view of Lemma 4.3.1, we have a stronger sufficient condition for the twist path $\mathbf{Q}=\mathbf{Q}(r)$ to be an extremal of the energy functional $\mathbb{E}$ is if it satisfies

$$
\frac{d}{dr}\left\{r^{n-1}F_\xi\left(r,1,\frac{n-1}{r^2}+|\dot{\mathbf{Q}}\theta|^2\right)\left[\dot{\mathbf{Q}}\theta\otimes\mathbf{Q}\theta-\mathbf{Q}\theta\otimes\dot{\mathbf{Q}}\theta\right]\right\}=0. \tag{4.28}
$$

From this we obtain the ODE system for $u=\mathbf{Q}\theta$,

$$
\begin{cases}
\dfrac{d}{dr}\left[r^{n-1}F_\xi\left(r,1,\dfrac{n-1}{r^2}+|\dot{\mathbf{Q}}\theta|^2\right)\right]=0 \quad \text{in } \mathbb{X}^n \\
\mathbf{Q}(a)=\mathbf{I}_n \\
\mathbf{Q}(b)=\mathbf{R}.
\end{cases}
\tag{4.29}
$$

4.3.3 The Case $\mathbf{Q}=\exp(\mathscr{G}\mathbf{H})$: $u=\exp(\mathscr{G}\mathbf{H})v(x)$

Proposition 4.3.3 (Key identities for $u=\exp(\mathscr{G}(r)\mathbf{H})v(x)$)**.** *Let $r=|x|$ and $\theta=x/|x|$. Suppose $u=\mathbf{Q}(r)v(x)$ is a spherical twist defined by a twice continuously differentiable twist path $\mathbf{Q}$, where $\mathbf{Q}(r)=\exp(\mathscr{G}(r)\mathbf{H})$ for some $\mathscr{G}\in\mathscr{C}^2([a,b],\mathbb{R}^n)$ and some constant skew-symmetric matrix $\mathbf{H}$. Then the following identities hold:*

(i) $\nabla u=\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}v\otimes\theta+\mathbf{Q}\nabla v$,

(ii) $|\nabla u|^2=\dot{\mathscr{G}}^2|\mathbf{H}v|^2+|\nabla v|^2+2\dot{\mathscr{G}}\langle\mathbf{H}v\otimes\theta,\nabla v\rangle$,

(iii) $\Delta u=\ddot{\mathscr{G}}\mathbf{H}\mathbf{Q}v+\mathbf{Q}\Delta v+\dot{\mathscr{G}}^2\mathbf{H}^2\mathbf{Q}v+\dot{\mathscr{G}}(n-1)r^{-1}\mathbf{H}\mathbf{Q}v+2\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\nabla v\theta$,

(iv) $\Delta_p u = |\nabla u|^{p-2}\Big[\ddot{\mathscr{G}}\mathbf{H}\mathbf{Q}v + \mathbf{Q}\Delta v + \dot{\mathscr{G}}^2\mathbf{H}^2\mathbf{Q}v + \dot{\mathscr{G}}(n-1)r^{-1}\mathbf{H}\mathbf{Q}v + 2\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\nabla v\theta\Big]$

$$+ |\nabla u|^{p-4}\frac{p-2}{2}\Big[\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}v\otimes\theta + \mathbf{Q}\nabla v\Big]\Big[2\ddot{\mathscr{G}}\dot{\mathscr{G}}|\mathbf{H}v|^2\theta + \nabla|\nabla v|^2 + \dot{\mathscr{G}}^2\nabla|\mathbf{H}v|^2 + 2\ddot{\mathscr{G}}\langle\mathbf{H}v\otimes\theta,\nabla v\rangle\theta + 2\dot{\mathscr{G}}\nabla\langle\mathbf{H}v\otimes\theta,\nabla v\rangle\Big],$$

(v) div $\left[F_\xi(r,1,|\nabla u|^2)\nabla u\right]$

$$= F_\xi\Big[\ddot{\mathscr{G}}\mathbf{H}\mathbf{Q}v + \mathbf{Q}\Delta v + \dot{\mathscr{G}}^2\mathbf{H}^2\mathbf{Q}v + \dot{\mathscr{G}}(n-1)r^{-1}\mathbf{H}\mathbf{Q}v + 2\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\nabla v\theta\Big] + F_{\xi r}\Big[\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}v + \mathbf{Q}\nabla v\theta\Big] + F_{\xi\xi}\Big[\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}v\otimes\theta + \mathbf{Q}\nabla v\Big]\Big[2\ddot{\mathscr{G}}\dot{\mathscr{G}}|\mathbf{H}v|^2\theta + \nabla|\nabla v|^2 + \dot{\mathscr{G}}^2\nabla|\mathbf{H}v|^2 + 2\ddot{\mathscr{G}}\langle\mathbf{H}v\otimes\theta,\nabla v\rangle\theta + 2\dot{\mathscr{G}}\nabla\langle\mathbf{H}v\otimes\theta,\nabla v\rangle\Big].$$

Proof. We have $u = \exp(\mathscr{G}(r)\mathbf{H})v(x)$, so simply taking the gradient we have the first identity. The third identity quickly follows when we take divergence of this. For the second identity we again apply the Hilbert-Schmidt norm for ∇u as found for the first identity, indeed

$$\begin{aligned}|\nabla u|^2 &= \mathrm{tr}\left([\nabla u]\,[\nabla u]^t\right)\\ &= \dot{\mathscr{G}}^2\mathrm{tr}(\mathbf{H}\theta\otimes\theta\mathbf{H}^t) + \mathrm{tr}\left([\nabla v][\nabla v]^t\right)\\ &\quad + \mathrm{tr}(-\dot{\mathscr{G}}\mathbf{H}\nabla v\theta\otimes v) + \mathrm{tr}(\dot{\mathscr{G}}\mathbf{H}v\otimes\theta(\nabla v)^t)\\ &= \dot{\mathscr{G}}^2|\mathbf{H}v|^2 + |\nabla v|^2 + 2\dot{\mathscr{G}}\mathrm{tr}(\mathbf{H}v\otimes\theta(\nabla v)^t).\end{aligned}\tag{4.30}$$

The second identity follows when noting $\mathrm{tr}(B^TA) = \langle A,B\rangle$. The fifth identity can be written as

$$\mathrm{div}\,\left[F_\xi(r,1,|\nabla u|^2)\nabla u\right] = F_\xi\Delta u + \nabla u\left(F_{\xi r}\theta + F_{\xi\xi}\nabla|\nabla u|^2\right).\tag{4.31}$$

Using the second identity, we have $\nabla|\nabla u|^2$ as

$$\begin{aligned}\nabla|\nabla u|^2 &= 2\ddot{\mathscr{G}}\dot{\mathscr{G}}|\mathbf{H}v|^2\theta + \nabla|\nabla v|^2 + \dot{\mathscr{G}}^2\nabla|\mathbf{H}v|^2 + 2\ddot{\mathscr{G}}\langle\mathbf{H}v\otimes\theta,\nabla v\rangle\theta\\ &\quad + 2\dot{\mathscr{G}}\nabla\langle\mathbf{H}v\otimes\theta,\nabla v\rangle.\end{aligned}\tag{4.32}$$

Substituting the relevant identities with (4.32) into (4.31) gives the result. Again we take the special case $F(r,s,\xi) = 2p^{-1}\xi^{p/2}$ in the fifth identity to obtain the fourth. □

In the even dimensional case, we the $\mathbb{F}$-energy for $u = \exp(\mathscr{G}(r)\mathbf{H})v(x)$. In even dimensions we have $|\mathbf{H}v|^2 = 1$, and so

$$\begin{aligned}\mathbb{F}[u;\mathbb{X}^n] &= \int_{\mathbb{X}^n} F(|x|,1,|\nabla u|^2)\,dx\\ &= \int_a^b\int_{\mathbb{S}^{n-1}} F\left(r,1,\dot{\mathscr{G}}^2 + |\nabla v|^2 + 2\dot{\mathscr{G}}\langle\mathbf{H}v\otimes\theta,\nabla v\rangle\right)r^{n-1}\,d\mathcal{H}^{n-1}(\theta)dr\\ &= \int_a^b K(r,v,\nabla v,\dot{\mathscr{G}})r^{n-1}\,dr =: \mathbb{K}[v,\mathscr{G};(a,b)]\end{aligned}\tag{4.33}$$

where the integrand $K = K(r, g, \mathbf{B}, k) \in \mathscr{C}^{1,2,1,1}([a,b] \times \mathbb{S}^{n-1} \times \mathrm{T}\mathbb{S}^{n-1} \times \mathbb{R})$ is given by

$$K(r, g, \mathbf{B}, k) := \int_{\mathbb{S}^{n-1}} F\left(r, 1, k^2 + |\mathbf{B}|^2 + 2k\langle \mathbf{H}g \otimes \theta, \mathbf{B}\rangle\right) \, d\mathcal{H}^{n-1}(\theta). \tag{4.34}$$

Lemma 4.3.2. *In even dimensions, the Euler-Lagrange equation associated with the energy $\mathbb{K}$ defined by (4.33) is given by*

$$\frac{d}{dr}\left[F_\xi r^{n-1}\left(\dot{\mathscr{G}} + \langle \mathbf{H}v \otimes \theta, \nabla v\rangle\right)\right] = 0, \tag{4.35}$$

where $\dot{\mathscr{G}}$ denotes the derivative of $\mathscr{G}(r) \in \mathscr{C}^2([a,b], \mathbb{R})$, F_ξ denotes the derivative in the third variable of $F(r, 1, \dot{\mathscr{G}}^2 + |\nabla v|^2 + 2\dot{\mathscr{G}}\langle \mathbf{H}v \otimes \theta, \nabla v\rangle)$, $\mathbf{H}$ is a constant skew-symmetric matrix and $v(x) \in \mathscr{C}^2(\mathbb{X}^n, \mathbb{S}^{n-1})$, and $\theta = x|x|^{-1}$. From this we obtain the ODE system for $u(x) = \exp(\mathscr{G}(r)\mathbf{H})v(x)$,

$$\begin{cases} \frac{d}{dr}\left[F_\xi r^{n-1}\left(\dot{\mathscr{G}} + \langle \mathbf{H}v \otimes \theta, \nabla v\rangle\right)\right] = 0 & \text{in } \mathbb{X}^n \\ \mathscr{G}(a) = 0 \\ \mathscr{G}(b) = 2\pi k \\ v(a) = v(b) = 1. \end{cases} \tag{4.36}$$

Using Proposition 4.3.3, we have that for $u = \exp(\mathscr{G}\mathbf{H})v$, the operator $\mathscr{L}_C[u]$ looks like

$$\begin{aligned} \mathscr{L}_C[u] = {} & F_\xi\left[\ddot{\mathscr{G}}\mathbf{H}\mathbf{Q}v + \mathbf{Q}\Delta v + \dot{\mathscr{G}}^2\mathbf{H}^2\mathbf{Q}v + \dot{\mathscr{G}}(n-1)r^{-1}\mathbf{H}\mathbf{Q}v + 2\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\nabla v\theta\right] \\ & + F_{\xi r}\left[\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}v + \mathbf{Q}\nabla v\theta\right] + F_{\xi\xi}\left[\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}v \otimes \theta + \mathbf{Q}\nabla v\right]\left[2\ddot{\mathscr{G}}\dot{\mathscr{G}}|\mathbf{H}v|^2\theta\right. \\ & \left. + \nabla|\nabla v|^2 + \dot{\mathscr{G}}^2\nabla|\mathbf{H}v|^2 + 2\ddot{\mathscr{G}}\langle \mathbf{H}v \otimes \theta, \nabla v\rangle\theta + 2\dot{\mathscr{G}}\nabla\langle \mathbf{H}v \otimes \theta, \nabla v\rangle\right] \\ & + F_\xi\left[\dot{\mathscr{G}}^2|\mathbf{H}v|^2 + |\nabla v|^2 + 2\dot{\mathscr{G}}\langle \mathbf{H}v \otimes \theta, \nabla v\rangle\right]\mathbf{Q}v. \end{aligned} \tag{4.37}$$

Specialising to even dimensions so that $|\mathbf{H}v|^2 = 1$ and $\mathbf{H}^2 = -\mathbf{I}_n$, we can simplify this to

$$\begin{aligned} \mathscr{L}_C[u] = {} & F_\xi\left[\ddot{\mathscr{G}}\mathbf{H}\mathbf{Q}v + \mathbf{Q}\Delta v + \dot{\mathscr{G}}(n-1)r^{-1}\mathbf{H}\mathbf{Q}v + 2\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\nabla v\theta\right] \\ & + F_{\xi r}\left[\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}v + \mathbf{Q}\nabla v\theta\right] + F_{\xi\xi}\left[\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}v \otimes \theta + \mathbf{Q}\nabla v\right]\left[2\ddot{\mathscr{G}}\dot{\mathscr{G}}\theta\right. \\ & \left. + \nabla|\nabla v|^2 + 2\ddot{\mathscr{G}}\langle \mathbf{H}v \otimes \theta, \nabla v\rangle\theta + 2\dot{\mathscr{G}}\nabla\langle \mathbf{H}v \otimes \theta, \nabla v\rangle\right] \\ & + F_\xi\left[|\nabla v|^2 + 2\dot{\mathscr{G}}\langle \mathbf{H}v \otimes \theta, \nabla v\rangle\right]\mathbf{Q}v. \end{aligned} \tag{4.38}$$

4.3.4 The Case $\mathbf{Q} = \exp(\mathscr{G}\mathbf{H})$: $u = \exp(\mathscr{G}\mathbf{H})\theta$

Proposition 4.3.4 (Key identities for $u = \exp(\mathscr{G}(r)\mathbf{H})\theta$). *Let $r = |x|$ and $\theta = x/|x|$. Suppose $u = \mathbf{Q}(r)\theta$ is a spherical twist defined by a twice continuously differentiable twist path $\mathbf{Q}$, where $\mathbf{Q}(r) = \exp(\mathscr{G}(r)\mathbf{H})$ for some $\mathscr{G} \in \mathscr{C}^2([a,b], \mathbb{R}^n)$ and some constant skew-symmetric matrix $\mathbf{H}$. Then the following identities hold:*

(i) $\nabla u = \dot{\mathscr{G}}\mathbf{HQ}\theta \otimes \theta + \mathbf{Q}\left(\mathbf{I}_n - \theta \otimes \theta\right) r^{-1}$,

(ii) $|\nabla u|^2 = \dot{\mathscr{G}}^2 |\mathbf{H}\theta|^2 + (n-1)r^{-2}$,

(iii) $\Delta u = \ddot{\mathscr{G}}\mathbf{HQ}\theta + \dot{\mathscr{G}}^2\mathbf{H}^2\mathbf{Q}\theta + \dot{\mathscr{G}}(n-1)r^{-1}\mathbf{HQ}\theta - (n-1)r^{-2}\mathbf{Q}\theta$,

(iv)
$$\begin{aligned}\Delta_p u = {} & |\nabla u|^{p-2}\left[\ddot{\mathscr{G}}\mathbf{HQ}\theta + \dot{\mathscr{G}}^2\mathbf{H}^2\mathbf{Q}\theta + \dot{\mathscr{G}}(n-1)r^{-1}\mathbf{HQ}\theta - (n-1)r^{-2}\mathbf{Q}\theta\right] \\ & + |\nabla u|^{p-4}\frac{p-2}{2}\left[\ddot{\mathscr{G}}|\mathbf{H}\theta|^2\theta + \dot{\mathscr{G}}\nabla|\mathbf{H}\theta|^2 - 2(n-1)r^{-3}\theta\right] \\ & \times \left[\dot{\mathscr{G}}\mathbf{HQ}\theta \otimes \theta + \mathbf{Q}\left(\mathbf{I}_n - \theta \otimes \theta\right) r^{-1}\right].\end{aligned}$$

(v)
$$\begin{aligned}\operatorname{div} & \left[F_\xi(r,1,|\nabla u|^2)\nabla u\right] \\ & = F_\xi\left[\ddot{\mathscr{G}}\mathbf{HQ}\theta + \dot{\mathscr{G}}^2\mathbf{H}^2\mathbf{Q}\theta + \dot{\mathscr{G}}(n-1)r^{-1}\mathbf{HQ}\theta - (n-1)r^{-2}\mathbf{Q}\theta\right] \\ & \quad + F_{\xi r}\left[\dot{\mathscr{G}}\mathbf{HQ}\theta\right] + F_{\xi\xi}\left[\ddot{\mathscr{G}}|\mathbf{H}\theta|^2\theta + \dot{\mathscr{G}}\nabla|\mathbf{H}\theta|^2 - 2(n-1)r^{-3}\theta\right] \\ & \quad \times \left[\dot{\mathscr{G}}\mathbf{HQ}\theta \otimes \theta + \mathbf{Q}\left(\mathbf{I}_n - \theta \otimes \theta\right) r^{-1}\right].\end{aligned}$$

Using (4.3.4), we derive the constrained operator $\mathscr{L}_C$ in this case as

$$\begin{aligned}\mathscr{L}_C[u] = {} & F_\xi\left[\ddot{\mathscr{G}}\mathbf{HQ}\theta + \dot{\mathscr{G}}^2\mathbf{H}^2\mathbf{Q}\theta + \dot{\mathscr{G}}(n-1)r^{-1}\mathbf{HQ}\theta - (n-1)r^{-2}\mathbf{Q}\theta\right] \\ & + F_{\xi r}\left[\dot{\mathscr{G}}\mathbf{HQ}\theta\right] + F_{\xi\xi}\left[\ddot{\mathscr{G}}|\mathbf{H}\theta|^2\theta + \dot{\mathscr{G}}\nabla|\mathbf{H}\theta|^2 - 2(n-1)r^{-3}\theta\right] \\ & \times \left[\dot{\mathscr{G}}\mathbf{HQ}\theta \otimes \theta + \mathbf{Q}\left(\mathbf{I}_n - \theta \otimes \theta\right) r^{-1}\right] + F_\xi\left[\dot{\mathscr{G}}^2|\mathbf{H}\theta|^2 + (n-1)r^{-2}\right]\mathbf{Q}\theta. \qquad (4.39)\end{aligned}$$

In even dimensions we have $|\mathbf{H}\theta|^2 = 1$ and $\mathbf{H}^2 = -\mathbf{I}_n$, so (4.39) reduces to

$$\begin{aligned}\mathscr{L}_C[u] = {} & F_\xi\left[\ddot{\mathscr{G}}\mathbf{HQ}\theta - \dot{\mathscr{G}}^2\mathbf{Q}\theta + \dot{\mathscr{G}}(n-1)r^{-1}\mathbf{HQ}\theta - (n-1)r^{-2}\mathbf{Q}\theta\right] \\ & + F_{\xi r}\left[\dot{\mathscr{G}}\mathbf{HQ}\theta\right] + F_{\xi\xi}\left[\ddot{\mathscr{G}} - 2(n-1)r^{-3}\right]\left[\dot{\mathscr{G}}\mathbf{HQ}\theta\right] \\ & + F_\xi\left[\dot{\mathscr{G}}^2 + (n-1)r^{-2}\right]\mathbf{Q}\theta \\ = {} & \left[F_\xi\left(\ddot{\mathscr{G}} + \dot{\mathscr{G}}(n-1)r^{-1}\right) + F_{\xi r}\dot{\mathscr{G}} + F_{\xi\xi}\left(\ddot{\mathscr{G}}\dot{\mathscr{G}} - 2\dot{\mathscr{G}}(n-1)r^{-3}\right)\right]\mathbf{HQ}\theta \\ = {} & \frac{\mathbf{HQ}\theta}{r^{n-1}}\left(F_\xi\frac{d}{dr}\left[r^{n-1}\dot{\mathscr{G}}\right] + r^{n-1}\dot{\mathscr{G}}\left\{F_{\xi r}\frac{d}{dr}r + F_{\xi\xi}\frac{d}{dr}\left[\dot{\mathscr{G}}^2 + (n-1)r^{-2}\right]\right\}\right) \\ = {} & \frac{\mathbf{HQ}\theta}{r^{n-1}}\frac{d}{dr}\left[F_\xi r^{n-1}\dot{\mathscr{G}}\right]. \qquad (4.40)\end{aligned}$$

We can write the $\mathbb{F}$-energy for $u = \exp(\mathscr{G}(r)\mathbf{H})\theta$ as

$$\begin{aligned}\mathbb{F}[u;\mathbb{X}^n] & = \int_{\mathbb{X}^n} F(|x|,1,|\nabla u|^2)\,dx \\ & = \int_a^b \int_{\mathbb{S}^{n-1}} F\left(r,1,\dot{\mathscr{G}}^2 + (n-1)r^{-2}\right) r^{n-1}\,d\mathcal{H}^{n-1}(\theta)dr \\ & = \omega_{n-1}\int_a^b P(r,\dot{\mathscr{G}})r^{n-1}\,dr =: \mathbb{P}[\mathscr{G};(a,b)] \qquad (4.41)\end{aligned}$$

where the integrand $P(r,g) \in \mathscr{C}^2([a,b]\times\mathbb{R})$ is given by

$$P(r,g) := F\left(r, 1, g^2 + (n-1)r^{-2}\right) . \tag{4.42}$$

Lemma 4.3.3. *In even dimensions, and with F_ξ denoting the derivative in the third variable of $F(r,1,\dot{\mathscr{G}}^2+(n-1)r^{-2})$ and $\dot{\mathscr{G}}$ denoting the derivative of $\mathscr{G}(r)\in\mathscr{C}^2([a,b],\mathbb{R})$, the Euler-Lagrange equation associated with the energy $\mathbb{P}$ defined by (4.41) is given by*

$$\frac{d}{dr}\left[F_\xi r^{n-1}\dot{\mathscr{G}}\right] = 0 \tag{4.43}$$

From this we obtain the ODE system for $u = \exp(\mathscr{G}(r)\mathbf{H})\theta$,

$$\begin{cases} \dfrac{d}{dr}\left[F_\xi r^{n-1}\dot{\mathscr{G}}\right] = 0 & \text{in } \mathbb{X}^n \\ \mathscr{G}(a) = 0 \\ \mathscr{G}(b) = 2\pi k, k \in \mathbb{Z}^+. \end{cases} \tag{4.44}$$

Proof. Consider the variation on $\mathscr{G}$ given by $\mathscr{G}_\varepsilon = \mathscr{G} + \varepsilon\mathscr{H}$ for $\mathscr{H} \in \mathscr{C}_0^\infty(]a,b[)$ and $\varepsilon \in \mathbb{R}$. We differentiate the energy $\mathbb{P}[\mathscr{G}_\varepsilon;(a,b)]$ with respect to ε and set ε to zero as follows.

$$\begin{aligned} 0 = \frac{d}{d\varepsilon}\mathbb{P}[\mathscr{G}_\varepsilon;(a,b)]\bigg|_{\varepsilon=0} &= \frac{d}{d\varepsilon}\int_a^b P(r,\mathscr{G}_\varepsilon)r^{n-1}\,dr\bigg|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon}\int_a^b F\left(r,1,\dot{\mathscr{G}}_\varepsilon^2+(n-1)r^{-2}\right)\omega_{n-1}r^{n-1}\,dr\bigg|_{\varepsilon=0} \\ &= \int_a^b F_\xi \frac{d}{d\varepsilon}\left[\dot{\mathscr{G}}_\varepsilon^2 + (n-1)r^{-2}\right] r^{n-1}\,dr\bigg|_{\varepsilon=0} \\ &= \int_a^b 2F_\xi\dot{\mathscr{G}}\dot{\mathscr{H}}r^{n-1}\,dr \\ &= \int_a^b -\frac{d}{dr}\left[F_\xi r^{n-1}\dot{\mathscr{G}}\right]\mathscr{H}\,dr. \end{aligned} \tag{4.45}$$

As this holds for all $\mathscr{H} \in \mathscr{C}_0^\infty(]a,b[)$, we have the result. □

Applying Lemma 4.3.3 to (4.40), we see that $\mathscr{L}_C[u = \exp(\mathscr{G}\mathbf{H})\theta] = 0$.

4.4 Specialised Twist Paths: The Unconstrained Case

In this section we consider the unconstrained problem by taking $u(x) = f(r)\mathbf{Q}v(x)$ so that $|u| = |f|$. We then specialise to functions $u = f\mathbf{Q}\theta$, $\theta = x|x|^{-1}$, and then further to when $\mathbf{Q} = \exp(\mathscr{G}(r)\mathbf{H})$, for $\mathscr{G} \in \mathscr{C}^2([a,b],\mathbb{R}^n)$.

By referring to the calculations in Proposition 4.2.1 we can write the $\mathbb{F}$-energy of a spherical twist $u = f\mathbf{Q}v$ as

$$\begin{aligned}
\mathbb{F}[u;\mathbb{X}^n] &= \int_{\mathbb{X}^n} F(|x|, |u|^2, |\nabla u|^2)\,dx \\
&= \int_a^b \int_{\mathbb{S}^{n-1}} F\left(r, f^2, \begin{array}{c} 2f^2\langle \mathbf{Q}^t\dot{\mathbf{Q}}v \otimes \theta, \nabla v\rangle + \dot{f}^2 \\ +f^2|\mathbf{Q}^t\dot{\mathbf{Q}}v|^2 + f^2|\nabla v|^2 \end{array}\right) r^{n-1}\,d\mathcal{H}^{n-1}(\theta)dr \\
&= \int_a^b L(r, \mathbf{Q}^t\dot{\mathbf{Q}}, f, \dot{f}, v, \nabla v) r^{n-1}\,dr =: \mathbb{L}[\mathbf{Q}, f, v; (a,b)] \qquad (4.46)
\end{aligned}$$

where the integrand $L = L(r, \mathbf{A}, g, h, k, \mathbf{B}) \in \mathscr{C}^{1,2,1,2,1,2}([a,b]\times\mathbb{R}^{n\times n}\times\mathbb{R}\times\mathbb{R}\times\mathbb{S}^{n-1}\times\mathrm{T}\mathbb{S}^{n-1})$ is given by

$$L(r, \mathbf{A}, g, h, k, \mathbf{B}) := \int_{\mathbb{S}^{n-1}} F\left(r, g^2, \begin{array}{c} 2g^2\langle \mathbf{A}k \otimes \theta, \mathbf{B}\rangle + h^2 \\ +g^2|\mathbf{B}k|^2 + g^2|\mathbf{B}|^2 \end{array}\right) d\mathcal{H}^{n-1}(\theta). \qquad (4.47)$$

Lemma 4.4.1 (Unconstrained Euler-Lagrange equation)**.** *The Euler-Lagrange equation associated with the energy* $\mathbb{L}$ *defined by* (4.46) *over the space of admissible twist paths* $\mathscr{B}^p_{\mathbf{R}}$ *is given by*

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} &\operatorname{div}\left[f^2F_\xi r^{n-1}\left(\mathbf{Q}^t\dot{\mathbf{Q}}v \otimes \theta + \nabla v\right)\right] \\
&\qquad + f^2F_\xi(\nabla v)^t\mathbf{Q}^t\dot{\mathbf{Q}}(\mathbf{I}_n - v\otimes v)\theta d\mathcal{H}^{n-1}(\theta) = 0. \qquad (4.48)
\end{aligned}$$

Proof. For real ε, consider the variation on $f = f(r)$ defined by $f_\varepsilon = f + \varepsilon h$ for $h \in \mathscr{C}^\infty_0(]a,b[)$.

$$\begin{aligned}
0 &= \frac{d}{d\varepsilon}\mathbb{L}[\mathbf{Q}, f_\varepsilon, v; (a,b)]\Big|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_a^b L(r, \mathbf{Q}^t\dot{\mathbf{Q}}, f_\varepsilon, \dot{f}_\varepsilon, v, \nabla v) r^{n-1}\,dr\Big|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\int_a^b \int_{\mathbb{S}^{n-1}} F\left(r, f_\varepsilon^2, \begin{array}{c} 2f_\varepsilon^2\langle \mathbf{Q}^t\dot{\mathbf{Q}}v \otimes \theta, \nabla v\rangle + \dot{f}_\varepsilon^2 \\ +f_\varepsilon^2|\mathbf{Q}^t\dot{\mathbf{Q}}v|^2 + f_\varepsilon^2|\nabla v|^2 \end{array}\right) d\mathcal{H}^{n-1}(\theta) r^{n-1}\,dr\Big|_{\varepsilon=0} \\
&= \int_a^b \int_{\mathbb{S}^{n-1}} F_s\frac{d}{d\varepsilon}\left[f_\varepsilon^2\right] d\mathcal{H}^{n-1}(\theta) r^{n-1}\,dr\Big|_{\varepsilon=0} \\
&+ \int_a^b \int_{\mathbb{S}^{n-1}} F_\xi\frac{d}{d\varepsilon}\left[\begin{array}{c} 2f_\varepsilon^2\langle \mathbf{Q}^t\dot{\mathbf{Q}}v \otimes \theta, \nabla v\rangle + \dot{f}_\varepsilon^2 \\ +f_\varepsilon^2|\mathbf{Q}^t\dot{\mathbf{Q}}v|^2 + f_\varepsilon^2|\nabla v|^2 \end{array}\right] d\mathcal{H}^{n-1}(\theta) r^{n-1}\,dr\Big|_{\varepsilon=0} \\
&= \int_a^b \int_{\mathbb{S}^{n-1}} 2F_sfh + F_\xi\left(\begin{array}{c} 4fh\langle \mathbf{Q}^t\dot{\mathbf{Q}}v \otimes \theta, \nabla v\rangle + 2\dot{f}\dot{h} \\ +2fh|\mathbf{Q}^t\dot{\mathbf{Q}}v|^2 + 2fh|\nabla v|^2 \end{array}\right) d\mathcal{H}^{n-1}(\theta) r^{n-1}\,dr. \qquad (4.49)
\end{aligned}$$

This lets us write the Euler-Lagrange equation as

$$\int_{\mathbb{S}^{n-1}} \frac{d}{dr}\left[F_\xi r^{n-1}\dot{f}\right] - \left[F_sf + F_\xi f\left(2\langle \mathbf{Q}^t\dot{\mathbf{Q}}v \otimes \theta, \nabla v\rangle + |\mathbf{Q}^t\dot{\mathbf{Q}}v|^2 + |\nabla v|^2\right)\right] d\mathcal{H}^{n-1}(\theta) = 0 \qquad (4.50)$$

Now consider the variation on $\mathbf{Q}$ defined as $\mathbf{Q}_\varepsilon = \mathbf{Q} + \varepsilon(\mathbf{F} - \mathbf{F}^t)\mathbf{Q}$, with $\mathbf{F} \in \mathscr{C}_0^\infty((a,b), \mathbb{M}^{n\times n})$ arbitrary:

$$
\begin{aligned}
0 &= \frac{d}{d\varepsilon}\mathbb{L}[\mathbf{Q}_\varepsilon, f, v; (a,b)]\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_a^b L(r, \mathbf{Q}_\varepsilon^t\dot{\mathbf{Q}}_\varepsilon, f, \dot f, v, \nabla v) r^{n-1}\, dr\bigg|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\int_a^b \int_{\mathbb{S}^{n-1}} F\left(r, f^2, \begin{matrix} 2f^2\langle \mathbf{Q}_\varepsilon^t\dot{\mathbf{Q}}_\varepsilon v\otimes\theta, \nabla v\rangle + \dot f^2 \\ +f^2|\mathbf{Q}_\varepsilon^t\dot{\mathbf{Q}}_\varepsilon v|^2 + f^2|\nabla v|^2 \end{matrix}\right) r^{n-1}\, d\mathcal{H}^{n-1}(\theta) dr\bigg|_{\varepsilon=0} \\
&= \int_a^b \int_{\mathbb{S}^{n-1}} F_\xi \frac{d}{d\varepsilon}\Big(2f^2\langle \mathbf{Q}_\varepsilon^t\dot{\mathbf{Q}}_\varepsilon v\otimes\theta, \nabla v\rangle\Big) r^{n-1}\, d\mathcal{H}^{n-1}(\theta) dr\bigg|_{\varepsilon=0} \\
&\quad + \int_a^b \int_{\mathbb{S}^{n-1}} F_\xi \frac{d}{d\varepsilon}\Big(f^2|\mathbf{Q}_\varepsilon^t\dot{\mathbf{Q}}_\varepsilon v|^2\Big) r^{n-1}\, d\mathcal{H}^{n-1}(\theta) dr\bigg|_{\varepsilon=0} \\
&= \int_a^b \Big\langle 2\int_{\mathbb{S}^{n-1}} \frac{d}{dr}\Big[F_\xi r^{n-1} f^2 \dot{\mathbf{Q}} v\otimes\theta(\nabla v)^t\mathbf{Q}^t\Big]\, d\mathcal{H}^{n-1}(\theta), \mathbf{F} - \mathbf{F}^t\Big\rangle dr \\
&\quad + \int_a^b \Big\langle -2\int_{\mathbb{S}^{n-1}} \frac{d}{dr}\Big[F_\xi r^{n-1} f^2 \dot{\mathbf{Q}} v\otimes \mathbf{Q} v\Big]\, d\mathcal{H}^{n-1}(\theta), \mathbf{F} - \mathbf{F}^t\Big\rangle dr.
\end{aligned}
\tag{4.51}
$$

Since this holds for all $\mathbf{F} \in \mathscr{C}_0^\infty((a,b), \mathbb{M}^{n\times n})$, the skew-symmetric part of the integral over $\mathbb{S}^{n-1}$ must be zero, and we have

$$
\int_{\mathbb{S}^{n-1}} \frac{d}{dr}\begin{bmatrix} F_\xi r^{n-1} f^2\Big\{\dot{\mathbf{Q}} v\otimes\theta(\nabla v)^t\mathbf{Q}^t - \mathbf{Q}\nabla v\theta\otimes v\dot{\mathbf{Q}}^t\Big\} \\ +F_\xi r^{n-1} f^2\Big\{\dot{\mathbf{Q}} v\otimes\mathbf{Q} v - \mathbf{Q} v\otimes\dot{\mathbf{Q}} v\Big\} \end{bmatrix} d\mathcal{H}^{n-1}(\theta) = 0. \tag{4.52}
$$

Now consider the variation of v given by $v_\varepsilon = (v+\varepsilon h)|v+\varepsilon h|^{-1}$ for $h \in \mathscr{C}_0^\infty(\mathbb{X}^n, \mathbb{S}^{n-1})$:

$$
\begin{aligned}
0 &= \frac{d}{d\varepsilon}\mathbb{L}[\mathbf{Q}, f, v_\varepsilon; (a,b)]\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\int_a^b L(r, \mathbf{Q}^t\dot{\mathbf{Q}}, f, \dot f, v_\varepsilon, \nabla v_\varepsilon) r^{n-1}\, dr\bigg|_{\varepsilon=0} \\
&= \frac{d}{d\varepsilon}\int_a^b \int_{\mathbb{S}^{n-1}} F\left(r, f^2, \begin{matrix} 2f^2\langle \mathbf{Q}^t\dot{\mathbf{Q}} v_\varepsilon\otimes\theta, \nabla v_\varepsilon\rangle + \dot f^2 \\ +f^2|\mathbf{Q}^t\dot{\mathbf{Q}} v_\varepsilon|^2 + f^2|\nabla v_\varepsilon|^2 \end{matrix}\right) r^{n-1}\, d\mathcal{H}^{n-1}(\theta)\, dr\bigg|_{\varepsilon=0} \\
&= \int_a^b \int_{\mathbb{S}^{n-1}} f^2 F_\xi \frac{d}{d\varepsilon}\begin{pmatrix} 2\langle \mathbf{Q}^t\dot{\mathbf{Q}} v_\varepsilon\otimes\theta, \nabla v_\varepsilon\rangle \\ +|\mathbf{Q}^t\dot{\mathbf{Q}} v_\varepsilon|^2 + |\nabla v_\varepsilon|^2 \end{pmatrix} r^{n-1}\, d\mathcal{H}^{n-1}(\theta)\, dr\bigg|_{\varepsilon=0} \\
&= \int_a^b \int_{\mathbb{S}^{n-1}} 2f^2 F_\xi \begin{pmatrix} 2\langle \mathbf{Q}^t\dot{\mathbf{Q}}(\mathbf{I}_n - v\otimes v)h\otimes\theta, \nabla v\rangle \\ +2\langle \mathbf{Q}^t\dot{\mathbf{Q}} v\otimes\theta, \nabla h\rangle + 2\langle \nabla v, \nabla h\rangle \end{pmatrix} r^{n-1}\, d\mathcal{H}^{n-1}(\theta)\, dr \\
&= \int_a^b \int_{\mathbb{S}^{n-1}} 2f^2 F_\xi \Big(2\langle \mathbf{Q}^t\dot{\mathbf{Q}} v\otimes\theta + \nabla v, \nabla h\rangle\Big) r^{n-1}\, d\mathcal{H}^{n-1}(\theta)\, dr \\
&\quad + \int_a^b \int_{\mathbb{S}^{n-1}} 2f^2 F_\xi \Big(2\langle (\nabla v)^t\mathbf{Q}^t\dot{\mathbf{Q}}(\mathbf{I}_n - v\otimes v)\theta, h\rangle\Big) r^{n-1}\, d\mathcal{H}^{n-1}(\theta)\, dr.
\end{aligned}
\tag{4.53}
$$

Since this is true for all $h \in \mathscr{C}_0^\infty(\mathbb{X}^n, \mathbb{S}^{n-1})$, we can write

$$\int_{\mathbb{S}^{n-1}} \operatorname{div}\left[f^2 F_\xi r^{n-1}\left(\mathbf{Q}^t\dot{\mathbf{Q}}v \otimes \theta + \nabla v\right)\right] \tag{4.54}$$

$$+ f^2 F_\xi (\nabla v)^t \mathbf{Q}^t \dot{\mathbf{Q}}(\mathbf{I}_n - v \otimes v)\theta d\mathcal{H}^{n-1}(\theta) = 0. \tag{4.55}$$

□

From this we obtain the ODE system for $u = f\mathbf{Q}v$,

$$\begin{cases} \dfrac{d}{dr}\left[F_\xi r^{n-1}\dot{f}\right] = F_s f + F_\xi f\left(2\langle\dot{\mathbf{Q}}v \otimes \theta, \nabla v\mathbf{Q}\rangle + |\dot{\mathbf{Q}}v|^2 + |\nabla v|^2\right) & \text{in } \mathbb{X}^n, \\ \mathbf{Q}(a) = \mathbf{I}_n, & \\ \mathbf{Q}(b) = \mathbf{R}, & \\ f(a) = f(b) = 1 & \text{on } \partial\mathbb{X}^n. \end{cases} \tag{4.56}$$

Proposition 4.4.1 (Key identities for $u = f\mathbf{Q}\theta$)**.** *Let $r = |x|$ and $\theta = x/|x|$. Suppose $u = f\mathbf{Q}(r)\theta$ is a spherical twist defined by a twice continuously differentiable twist path $\mathbf{Q}$, and some function $f \in \mathscr{C}^2([a,b])$. Then the following identities hold:*

(i) $\nabla u = \dfrac{f}{r}\mathbf{Q} + \left(\dot{f} - \dfrac{f}{r}\right)\mathbf{Q}\theta \otimes \theta + f\dot{\mathbf{Q}}\theta \otimes \theta,$

(ii) $|\nabla u|^2 = f^2\left((n-1)r^{-2} + |\dot{\mathbf{Q}}\theta|^2\right) + \dot{f}^2,$

(iii) $\Delta u = f\ddot{\mathbf{Q}}\theta + \dot{\mathbf{Q}}\theta\left((n-1)fr^{-1} + 2\dot{f}\right) + \mathbf{Q}\theta\left(\ddot{f} + (n-1)\dot{f}r^{-1} - (n-1)fr^{-2}\right),$

(iv)
$$\begin{aligned} \Delta_p u = {} & |\nabla u|^{p-2}\left[f\ddot{\mathbf{Q}}\theta + \dot{\mathbf{Q}}\theta\left((n-1)fr^{-1} + 2\dot{f}\right) + \mathbf{Q}\theta\left(\ddot{f} + (n-1)\dot{f}r^{-1} - (n-1)fr^{-2}\right)\right] \\ & + \frac{p-2}{2}|\nabla u|^{p-4}\Big[\left(\dot{f}\mathbf{Q}\theta + f\dot{\mathbf{Q}}\theta\right)\left\{2\dot{f}\ddot{f} + 2f\dot{f}|\dot{\mathbf{Q}}\theta|^2 + 2f\dot{f}(n-1)r^{-1}\right\} \\ & + \left(fr^{-1}\mathbf{Q} + (\dot{f} - fr^{-1})\mathbf{Q}\theta \otimes \theta + f\dot{\mathbf{Q}}\theta \otimes \theta\right)\left\{2f\dot{f}\dot{\mathbf{Q}}\dot{\mathbf{Q}}\theta + 2f\dot{f}\mathbf{Q}\ddot{\mathbf{Q}}\theta + f^2\nabla|\dot{\mathbf{Q}}\theta|^2\right\}\Big]. \end{aligned}$$

(v)
$$\begin{aligned} & \operatorname{div}\left[F_\xi(r, |u|^2, |\nabla u|^2)\nabla u\right] \\ & = F_\xi\left[f\ddot{\mathbf{Q}}\theta + \dot{\mathbf{Q}}\theta\left((n-1)fr^{-1} + 2\dot{f}\right) + \mathbf{Q}\theta\left(\ddot{f} + (n-1)\dot{f}r^{-1} - (n-1)fr^{-2}\right)\right] \\ & + F_{\xi r}\left(f\dot{\mathbf{Q}}\theta + \mathbf{Q}\dot{f}\theta\right) + 2F_{\xi s}\dot{f}f\left(f\dot{\mathbf{Q}}\theta + \mathbf{Q}\dot{f}\theta\right) \\ & + F_{\xi\xi}\Big[\left(\dot{f}\mathbf{Q}\theta + f\dot{\mathbf{Q}}\theta\right)\left\{2\dot{f}\ddot{f} + 2f\dot{f}|\dot{\mathbf{Q}}\theta|^2 + 2f\dot{f}(n-1)r^{-1}\right\} \\ & + \left(fr^{-1}\mathbf{Q} + (\dot{f} - fr^{-1})\mathbf{Q}\theta \otimes \theta + f\dot{\mathbf{Q}}\theta \otimes \theta\right)\left\{2f\dot{f}\dot{\mathbf{Q}}\dot{\mathbf{Q}}\theta + 2f\dot{f}\mathbf{Q}\ddot{\mathbf{Q}}\theta + f^2\nabla|\dot{\mathbf{Q}}\theta|^2\right\}\Big]. \end{aligned}$$

Proof. We can prove the first and third identities by simple differentiation, first the gradient of u to obtain ∇u, then the divergence of this. For the second identity, we take the Hilbert-Schmidt norm of ∇u,

$$|\nabla u|^2 = \operatorname{tr}\left([\nabla u][\nabla u]^t\right),$$

where we are left with the result after some cancellation of terms. Taking the divergence of $F_\xi \nabla u$, where $F_\xi = F_\xi(r, |u|^2, |\nabla u|^2)$, we have

$$\operatorname{div}\left[F_\xi \nabla u\right] = F_\xi \Delta u + \nabla u \left(F_{\xi r}\theta + F_{\xi s}\nabla\left[|u|^2\right] + F_{\xi\xi}\nabla|\nabla u|^2\right). \tag{4.57}$$

Substituting $\nabla|u|^2 = \nabla f^2 = 2f\dot{f}\theta$, and

$$\nabla|\nabla u|^2 = 2f\dot{f}\left(|\dot{\mathbf{Q}}\theta|^2\theta + (n-1)r^{-1}\theta + \dot{\mathbf{Q}}\dot{\mathbf{Q}}\theta + \mathbf{Q}\ddot{\mathbf{Q}}\theta\right) + f^2\nabla|\dot{\mathbf{Q}}\theta|^2 + 2\dot{f}\ddot{f}\theta \tag{4.58}$$

into (4.57), together with the first identity for ∇u, gives the fifth identity. Once again we take the substitution $F(r,s,\xi) = 2\xi^{p/2}p^{-1}$ in the fifth identity to obtain the fourth. $\square$

The unconstrained operator $\mathscr{L}_U$ can then be written

$$\begin{aligned}
&\mathscr{L}_U[u = f\mathbf{Q}\theta] \\
&\quad = F_\xi\left[f\ddot{\mathbf{Q}}\theta + \dot{\mathbf{Q}}\theta\left((n-1)fr^{-1} + 2\dot{f}\right) + \mathbf{Q}\theta\left(\ddot{f} + (n-1)\dot{f}r^{-1} - (n-1)fr^{-2}\right)\right] \\
&\qquad + F_{\xi r}\left(f\dot{\mathbf{Q}}\theta + \mathbf{Q}\dot{f}\theta\right) + 2F_{\xi s}\dot{f}f\left(f\dot{\mathbf{Q}}\theta + \mathbf{Q}\dot{f}\theta\right) \\
&\qquad + F_{\xi\xi}\Big[\left(\dot{f}\mathbf{Q}\theta + f\dot{\mathbf{Q}}\theta\right)\left\{2\dot{f}\ddot{f} + 2f\dot{f}|\dot{\mathbf{Q}}\theta|^2 + 2f\dot{f}(n-1)r^{-1}\right\} \\
&\qquad + \left(fr^{-1}\mathbf{Q} + (\dot{f} - fr^{-1})\mathbf{Q}\theta\otimes\theta + f\dot{\mathbf{Q}}\theta\otimes\theta\right)\left\{2f\dot{f}\dot{\mathbf{Q}}\dot{\mathbf{Q}}\theta + 2f\dot{f}\mathbf{Q}\ddot{\mathbf{Q}}\theta + f^2\nabla|\dot{\mathbf{Q}}\theta|^2\right\}\Big].
\end{aligned} \tag{4.59}$$

With $|\nabla u|^2$ as in Proposition 4.4.1 we can write the $\mathbb{F}$-energy of a spherical twist $u = f\mathbf{Q}\theta$ as

$$\begin{aligned}
\mathbb{F}[u;\mathbb{X}^n] &= \int_{\mathbb{X}^n} F(|x|, |u|^2, |\nabla u|^2)\,dx \\
&= \int_a^b \int_{\mathbb{S}^{n-1}} F\left(r, f^2, f^2\left[(n-1)\,r^{-2} + |\dot{\mathbf{Q}}\theta|^2\right] + \dot{f}^2\right) r^{n-1}\,d\mathcal{H}^{n-1}(\theta)dr \\
&= \int_a^b Y(r, \dot{\mathbf{Q}}, f)r^{n-1}\,dr =: \mathbb{Y}[\mathbf{Q}, f; (a,b)]
\end{aligned} \tag{4.60}$$

where the integrand $Y = Y(r, \mathbf{A}, f)$ with $a \le r \le b$ and $\mathbf{A}$ in the space of $n\times n$ matrices is given by

$$L = \int_{\mathbb{S}^{n-1}} F\left(r, f^2, f^2\left[(n-1)\,r^{-2} + |\dot{\mathbf{Q}}\theta|^2\right] + \dot{f}^2\right) r^{n-1}\,d\mathcal{H}^{n-1}(\theta). \tag{4.61}$$

Proposition 4.4.2 (Key identities for $u = f\exp(\mathscr{G}(r)\mathbf{H})\theta$). *Let $u = f\exp(\mathscr{G}(r)\mathbf{H})\theta$, for some $f \in \mathscr{C}^2([a,b],\mathbb{R})$, $\mathscr{G} \in \mathscr{C}^2([a,b],\mathbb{R}^n)$, and some constant skew-symmetric matrix $\mathbf{H}$. Then the following identities hold:*

(i) $\nabla u = \left(\dot{f} - fr^{-1}\right)\mathbf{Q}\theta\otimes\theta + f\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta\otimes\theta + fr^{-1}\mathbf{Q}$,

(ii) $|\nabla u|^2 = f^2\dot{\mathscr{G}}^2|\mathbf{H}\theta|^2 + \dot{f}^2 + (n-1)f^2r^{-2}$,

(iii) $\Delta u = \ddot{f}\mathbf{Q}\theta + \dot{f}\left(2\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta + (n-1)r^{-1}\mathbf{Q}\theta\right)$
$$+ f\left(\ddot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta + \dot{\mathscr{G}}^2\mathbf{H}^2\mathbf{Q}\theta + (n-1)r^{-1}\left[\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta - r^{-1}\mathbf{Q}\theta\right]\right),$$

(iv) $\Delta_p u = |\nabla u|^{p-2}\Big[\ddot{f}\mathbf{Q}\theta + \dot{f}\left(2\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta + (n-1)r^{-1}\mathbf{Q}\theta\right)$
$$+ f\left(\ddot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta + \dot{\mathscr{G}}^2\mathbf{H}^2\mathbf{Q}\theta + (n-1)r^{-1}\left[\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta - r^{-1}\mathbf{Q}\theta\right]\right)\Big]$$
$$+ |\nabla u|^{p-4}\frac{p-2}{2}\left[\left(\dot{f} - fr^{-1}\right)\mathbf{Q}\theta\otimes\theta + f\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta\otimes\theta + fr^{-1}\mathbf{Q}\right]$$
$$\times\Big[2\ddot{f}\dot{f}\theta + 2f\dot{f}\dot{\mathscr{G}}^2|\mathbf{H}\theta|^2\theta + 2f^2\dot{\mathscr{G}}\ddot{\mathscr{G}}|\mathbf{H}\theta|^2\theta + f^2\dot{\mathscr{G}}^2\nabla|\mathbf{H}\theta|^2$$
$$+ 2(n-1)(f\dot{f}r^{-2}\theta - f^2r^{-3}\theta)\Big],$$

(v) $\operatorname{div}\left[F_\xi(r, f^2, |\nabla u|^2)\nabla u\right]$
$$= F_\xi\Big[\ddot{f}\mathbf{Q}\theta + \dot{f}\left(2\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta + (n-1)r^{-1}\mathbf{Q}\theta\right)$$
$$+ f\left(\ddot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta + \dot{\mathscr{G}}^2\mathbf{H}^2\mathbf{Q}\theta + (n-1)r^{-1}\left[\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta - r^{-1}\mathbf{Q}\theta\right]\right)\Big]$$
$$+ F_{\xi r}\left[\dot{f}\mathbf{Q}\theta + f\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta\right] + 2f\dot{f}F_{\xi s}\left[\dot{f}\mathbf{Q}\theta + f\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta\right]$$
$$+ F_{\xi\xi}\left[\left(\dot{f} - fr^{-1}\right)\mathbf{Q}\theta\otimes\theta + f\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta\otimes\theta + fr^{-1}\mathbf{Q}\right]$$
$$\times\Big[2\ddot{f}\dot{f}\theta + 2f\dot{f}\dot{\mathscr{G}}^2|\mathbf{H}\theta|^2\theta + 2f^2\dot{\mathscr{G}}\ddot{\mathscr{G}}|\mathbf{H}\theta|^2\theta + f^2\dot{\mathscr{G}}^2\nabla|\mathbf{H}\theta|^2$$
$$+ 2(n-1)(f\dot{f}r^{-2}\theta - f^2r^{-3}\theta)\Big].$$

Proof. Taking $u(x) = f(r)\exp(\mathscr{G}(r)\mathbf{H})\theta$, we take the gradient of this for the first identity, and then further we take the divergence of the first identity for the third identity, noting that $r = |x|$ and $\theta = x|x|^{-1}$, and that since $\mathbf{H}$ is skew-symmetric we have $\langle\mathbf{H}, \theta\otimes\theta\rangle = 0$. Taking the Hilbert-Schmidt norm of the first identity ∇u, we have

$$|\nabla u|^2 = \operatorname{tr}\left([\nabla u]\,[\nabla u]^t\right), \tag{4.62}$$

which when evaluated gives the result after some cancellation of terms. For the fifth identity we note first that $\nabla|u|^2 = \nabla f^2 = 2f\dot{f}\theta$, and that

$$\nabla|\nabla u|^2 = 2\ddot{f}\dot{f}\theta + 2f\dot{f}\dot{\mathscr{G}}^2|\mathbf{H}\theta|^2\theta + 2f^2\dot{\mathscr{G}}\ddot{\mathscr{G}}|\mathbf{H}\theta|^2\theta + f^2\dot{\mathscr{G}}^2\nabla|\mathbf{H}\theta|^2 \tag{4.63}$$
$$+ 2(n-1)(f\dot{f}r^{-2}\theta - f^2r^{-3}\theta). \tag{4.64}$$

Expanding $\operatorname{div}\left[F_\xi(r, f^2, |\nabla u|^2)\nabla u\right]$ as

$$\operatorname{div}\left[F_\xi\right] = \Delta u F_\xi + \nabla u\left(F_{\xi r}\theta + F_{\xi s}\nabla f^2 + F_{\xi\xi}\nabla|\nabla u|^2\right), \tag{4.65}$$

then substituting in the values above, together with previous identities, we have the result. The fourth identity then immediately follows when we take the special case $F(r, s, \xi) = 2p^{-1}\xi^{p/2}$ in the fifth identity. $\square$

We can again consider the $\mathbb{F}$-energy in the even dimensional case, so for $u = f(r)\exp(\mathscr{G}(r)\mathbf{H})\theta$, with $|\nabla u|^2$ as in (4.4.2), and the dimension n being even so that $|\mathbf{H}\theta|^2 = 1$, we have

$$\begin{aligned}\mathbb{F}[u;\mathbb{X}^n] &= \int_{\mathbb{X}^n} F(|x|,|u|^2,|\nabla u|^2)\,dx \\ &= \int_a^b \int_{\mathbb{S}^{n-1}} F\left(r, f^2, f^2\dot{\mathscr{G}}^2 + \dot{f}^2 + (n-1)f^2r^{-2}\right) r^{n-1}\,d\mathcal{H}^{n-1}(\theta)dr \\ &= \omega_{n-1}\int_a^b M(r,f,\mathscr{G})r^{n-1}\,dr =: \mathbb{M}[f,\mathscr{G};(a,b)],\end{aligned} \tag{4.66}$$

where the integrand $M = M(r,f,\mathscr{G})$ with $a \le r \le b$ is given by

$$M(r,f,\mathscr{G}) := F\left(r, f^2, f^2\dot{\mathscr{G}}^2 + \dot{f}^2 + (n-1)f^2r^{-2}\right). \tag{4.67}$$

Taking variations in both f and $\mathscr{G}$ of the energy $\mathbb{M}[f,\mathscr{G};(a,b)]$, we arrive at the following lemma.

Lemma 4.4.2. *Let $f = f(r) \in \mathscr{C}^2([a,b],\mathbb{R})$ and $\mathscr{G} \in \mathscr{C}([a,b],\mathbb{R}^n)$. For even dimensions n, the Euler-Lagrange equations associated with the energy $\mathbb{M}[f,\mathscr{G};(a,b)]$ defined by (4.66) are given by*

$$\frac{d}{dr}\left[F_\xi \dot{f} r^{n-1}\right] = f\left[F_\xi r^{n-1}(\dot{\mathscr{G}}^2 + (n-1)r^{-2}) + F_s r^{n-1}\right], \tag{4.68}$$

and

$$\frac{d}{dr}\left[r^{n-1}F_\xi f^2\dot{\mathscr{G}}\right] = 0, \tag{4.69}$$

where F_ξ and F_s denote the differentials in the third and second variables respectively of $F(r, f^2, f^2\dot{\mathscr{G}}^2 + \dot{f}^2 + (n-1)f^2r^{-2})$. The resulting ODE system is given by

$$\begin{cases} \dfrac{d}{dr}\left[F_\xi \dot{f} r^{n-1}\right] = F_\xi r^{n-1}\left(f\dot{\mathscr{G}}^2 + (n-1)fr^{-2}\right) + F_s f r^{n-1} & \textit{in } \mathbb{X}^n, \\ \dfrac{d}{dr}\left[r^{n-1}F_\xi f^2\dot{\mathscr{G}}\right] = 0 & \textit{in } \mathbb{X}^n, \\ f(a) = f(b) = 1 & \textit{on } \partial\mathbb{X}^n, \\ \mathscr{G}(a) = 0, \\ \mathscr{G}(b) = 2\pi k. \end{cases} \tag{4.70}$$

Using Proposition 4.4.2, we can write the unconstrained differential operator $\mathscr{L}_U$ as

$$\begin{aligned}\mathscr{L}_U[u] = \; & F_\xi\Big[\ddot{f}\mathbf{Q}\theta + \dot{f}\left(2\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta + (n-1)r^{-1}\mathbf{Q}\theta\right) \\ & + f\left(\ddot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta + \dot{\mathscr{G}}^2\mathbf{H}^2\mathbf{Q}\theta + (n-1)r^{-1}\left[\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta - r^{-1}\mathbf{Q}\theta\right]\right)\Big] \\ & + F_{\xi r}\left[\dot{f}\mathbf{Q}\theta + f\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta\right] + 2f\dot{f}F_{\xi s}\left[\dot{f}\mathbf{Q}\theta + f\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta\right] \\ & + F_{\xi\xi}\left[\left(\dot{f} - fr^{-1}\right)\mathbf{Q}\theta\otimes\theta + f\dot{\mathscr{G}}\mathbf{H}\mathbf{Q}\theta\otimes\theta + fr^{-1}\mathbf{Q}\right] \\ & \times\Big[2\ddot{f}\dot{f}\theta + 2f\dot{f}\dot{\mathscr{G}}^2|\mathbf{H}\theta|^2\theta + 2f^2\dot{\mathscr{G}}\ddot{\mathscr{G}}|\mathbf{H}\theta|^2\theta + f^2\dot{\mathscr{G}}^2\nabla|\mathbf{H}\theta|^2 && (4.71) \\ & + 2(n-1)(f\dot{f}r^{-2}\theta - f^2r^{-3}\theta)\Big] - fF_s\mathbf{Q}\theta. && (4.72)\end{aligned}$$

where we have written $\mathbf{Q} = \exp(\mathscr{G}(r)\mathbf{H})$. Specialising to the even dimensional case so that $\mathbf{H}^2 = -\mathbf{I}_n$, and $|\mathbf{H}\theta|^2 = 1$, we can write

$$\begin{aligned}\mathscr{L}_U[u] = & \Big[F_\xi\left(2\dot{f}\dot{\mathscr{G}} + f\ddot{\mathscr{G}} + f(n-1)r^{-1}\dot{\mathscr{G}}\right) + F_{\xi r}f\dot{\mathscr{G}} + 2F_{\xi s}f^2\dot{f}\dot{\mathscr{G}} \\ & + 2F_{\xi\xi}\left(f\dot{f}\ddot{f}\dot{\mathscr{G}} + f^2\dot{f}\dot{\mathscr{G}}^3 + f^3\dot{\mathscr{G}}^2\ddot{\mathscr{G}} + f\dot{\mathscr{G}}(n-1)(f\dot{f}r^{-2} - f^2r^{-3})\right)\Big]\mathbf{H}\mathbf{Q}\theta \\ & + \Big[F_\xi\left(\ddot{f} + \dot{f}(n-1)r^{-1} - f\dot{\mathscr{G}}^2 - f(n-1)r^{-2}\right) - F_sf \\ & + 2F_{\xi\xi}\left(\ddot{f}\dot{f}^2 + f\dot{f}^2\dot{\mathscr{G}}^2 + f^2\dot{f}\dot{\mathscr{G}}\ddot{\mathscr{G}} + (n-1)(f\dot{f}^2r^{-2} - \dot{f}f^2r^{-3})\right) \\ & + F_{\xi r}\dot{f} + 2F_{\xi s}f\dot{f}^2\Big]\mathbf{Q}\theta \\ = & \mathscr{J}_1(r)\mathbf{H}\mathbf{Q}\theta + \mathscr{J}_2(r)\mathbf{Q}\theta, && (4.73)\end{aligned}$$

where we have grouped the terms as coefficients of $\mathbf{HQ}\theta$ and $\mathbf{Q}\theta$. Considering the terms multiplying $\mathbf{HQ}\theta$ alone, we can reduce these to a single derivative. Indeed, recalling that $F_\xi, F_{\xi r}, F_{\xi s}$, and $F_{\xi\xi}$ are all derivatives of the function $F(r, f^2, f^2\dot{\mathscr{G}}^2 + \dot{f}^2 + (n-1)f^2r^{-2})$, with $f = f(r)$ and $\mathscr{G} = \mathscr{G}(r)$, we have

$$\begin{aligned}\mathscr{J}_1(r) = & \Big[F_\xi\left(2\dot{f}\dot{\mathscr{G}} + f\ddot{\mathscr{G}} + f(n-1)r^{-1}\dot{\mathscr{G}}\right) + F_{\xi r}f\dot{\mathscr{G}} + 2F_{\xi s}f^2\dot{f}\dot{\mathscr{G}} \\ & + 2F_{\xi\xi}\left(f\dot{f}\ddot{f}\dot{\mathscr{G}} + f^2\dot{f}\dot{\mathscr{G}}^3 + f^3\dot{\mathscr{G}}^2\ddot{\mathscr{G}} + f\dot{\mathscr{G}}(n-1)(f\dot{f}r^{-2} - f^2r^{-3})\right)\Big] \\ = & \frac{1}{fr^{n-1}}\Bigg(f^2\dot{\mathscr{G}}r^{n-1}\left\{F_{\xi r}\frac{d}{dr}r + F_{\xi s}\frac{d}{dr}[f^2] + F_{\xi\xi}\frac{d}{dr}\left[f^2\dot{\mathscr{G}}^2 + \dot{f}^2 + (n-1)f^2r^{-2}\right]\right\} \\ & + F_\xi\frac{d}{dr}\left[f^2\dot{\mathscr{G}}r^{n-1}\right]\Bigg) = \frac{1}{fr^{n-1}}\frac{d}{dr}\left[r^{n-1}F_\xi f^2\dot{\mathscr{G}}\right] = 0, && (4.74)\end{aligned}$$

where in the last equality we have used that this is zero from Lemma 4.4.2. Now considering just the terms multiplying $\mathbf{Q}\theta$, we can write

$$\begin{aligned}
\mathscr{J}_2(r) = &\Big[F_\xi\left(\ddot{f} + \dot{f}(n-1)r^{-1} - f\dot{\mathscr{G}}^2 - f(n-1)r^{-2}\right) - F_s f \\
&+ 2F_{\xi\xi}\left(\ddot{f}\dot{f}^2 + f\dot{f}^2\dot{\mathscr{G}}^2 + f^2\dot{f}\dot{\mathscr{G}}\ddot{\mathscr{G}} + (n-1)(f\dot{f}^2r^{-2} - \dot{f}f^2r^{-3})\right) \\
&+ F_{\xi r}\dot{f} + 2F_{\xi s}f\dot{f}^2\Big] \\
=&\frac{1}{r^{n-1}}\Bigg(F_\xi\frac{d}{dr}\left[\dot{f}r^{n-1}\right] - r^{n-1}\left[F_\xi\left\{f\dot{\mathscr{G}}^2 + f(n-1)r^{-2}\right\} + F_s f\right] \\
&+ \dot{f}r^{n-1}\left\{F_{\xi r}\frac{d}{dr}r + F_{\xi s}\frac{d}{dr}[f^2] + F_{\xi\xi}\frac{d}{dr}\left[f^2\dot{\mathscr{G}}^2 + \dot{f}^2 + (n-1)f^2r^{-2}\right]\right\}\Bigg) \\
=&\frac{1}{r^{n-1}}\left(\frac{d}{dr}\left[F_\xi\dot{f}r^{n-1}\right] - F_\xi r^{n-1}\left(f\dot{\mathscr{G}}^2 + (n-1)fr^{-2}\right) - F_s f r^{n-1}\right) = 0, \qquad (4.75)
\end{aligned}$$

where the last equality is from Lemma 4.4.2. Hence $\mathscr{L}_U[u = f\exp(\mathscr{G}\mathbf{H})\theta] = 0$.

4.5 As a Limiting Case of an Alternative Energy

In this section we consider an alternative energy with an additional term. With $u = f(r)\exp(\mathscr{G}(r)\mathbf{H})\theta$ as at the end of the previous section, we redefine the energy $\mathbb{F}(u;\mathbb{X}^n)$ as

$$\begin{aligned}
\mathbb{F}_\varepsilon[u;\mathbb{X}^n] &= \int_{\mathbb{X}^n} F(|x|,|u|^2,|\nabla u|^2) + \frac{1}{\varepsilon}W(|u|^2)\,dx \\
&= \omega_{n-1}\int_a^b\left[F\left(r,f^2,f^2\dot{\mathscr{G}}^2 + \dot{f}^2 + (n-1)f^2r^{-2}\right) + \frac{1}{\varepsilon}W(f^2)\right]r^{n-1}\,dr \\
&= \omega_{n-1}\int_a^b A(r,f,\dot{f},\dot{\mathscr{G}})r^{n-1}\,dr =: \mathbb{A}[f,\mathscr{G};(a,b)], \qquad (4.76)
\end{aligned}$$

where $A(r,g,h,k)$ is given by

$$A(r,g,h,k) = F\left(r,g^2,g^2k^2 + h^2 + (n-1)g^2r^{-2}\right) + \frac{1}{\varepsilon}W(g^2). \qquad (4.77)$$

Here $W = W(x) \in \mathscr{C}^1(\mathbb{R})$ is a function that is zero at $|x| = 1$ and positive elsewhere, $F = F(r,s,\xi)$ is coercive in that $F(r,s,\xi) \geq c_0 + c_1|\xi|^{p/2}$ with $c_1 > 0$, and $\varepsilon \in \mathbb{R}$ is a constant. By choosing a suitable F and W we can see that this is of the same form as (4.67), so for each $\varepsilon \in \mathbb{R}$ we have a solution $(f_\varepsilon, \mathscr{G}_\varepsilon)$ to a system of ODEs with the boundary conditions for any $m \in \mathbb{Z}$

$$\begin{cases}
\mathscr{G}_\varepsilon(a) = 0, \\
\mathscr{G}_\varepsilon(b) = 2\pi m, \\
f_\varepsilon(a) = f_\varepsilon(b) = 1.
\end{cases} \qquad (4.78)$$

Writing $u_\varepsilon = f_\varepsilon\exp(\mathscr{G}_\varepsilon\mathbf{H})\theta$, we then have that u_ε solves the relevant unconstrained PDE problem $\mathscr{L}_U[u_\varepsilon] = 0$. Picking a suitable positive constant C, independent of ε, we can bound the energy $\mathbb{F}_\varepsilon[u_\varepsilon]$ as

$$0 \leq \int_a^b \left[F + \frac{1}{\varepsilon}W\right] r^{n-1}\, dr \leq C, \tag{4.79}$$

where $F = F(r, f_\varepsilon^2, f_\varepsilon^2 \dot{\mathscr{G}}_\varepsilon^2 + \dot{f}_\varepsilon^2 + (n-1)f_\varepsilon^2 r^{-2})$ and $W = W(f_\varepsilon^2)$. Noting that $W(x) \geq 0$, and recalling that F is coercive, we can write

$$0 \leq \int_a^b |f_\varepsilon^2 \dot{\mathscr{G}}_\varepsilon^2 + \dot{f}_\varepsilon^2 + (n-1)f_\varepsilon^2 r^{-2}|^{p/2} r^{n-1} dr \leq \int_a^b F r^{n-1}\, dr \leq C, \tag{4.80}$$

which leads to the inequalites

$$\sup_\varepsilon \int_a^b |\dot{\mathscr{G}}_\varepsilon|^p r^{n-1} dr \leq C, \quad \sup_\varepsilon \int_a^b |\dot{f}_\varepsilon|^p r^{n-1} dr \leq C. \tag{4.81}$$

Together with the boundary conditions given in (4.78), this implies boundedness of $\mathscr{G}_\varepsilon$ and f_ε with respect to the $\mathscr{W}^{1,p}([a,b])$ norm. That is, we have

$$\sup_\varepsilon ||\mathscr{G}_\varepsilon||_{1,p} < \infty, \quad \sup_\varepsilon ||f_\varepsilon||_{1,p} < \infty. \tag{4.82}$$

This in turn implies the existence of f and $\mathscr{G}$ such that $f_\varepsilon \rightharpoonup f$ and $\mathscr{G}_\varepsilon \rightharpoonup \mathscr{G}$ weakly in $\mathscr{W}^{1,p}([a,b])$.

With $W(x)$ as defined above, we can easily bound it as $W(x) \geq d|1-x|$ for some positive constant d. With $F \geq 0$ this gives

$$0 \leq d \int_a^b \left|1 - |f_\varepsilon|^2\right| r^{n-1} dr \leq \int_a^b W(|f_\varepsilon|^2) r^{n-1}\, dr \leq \varepsilon C, \tag{4.83}$$

so that in the limit we must have $f_\varepsilon \to 1$ *a.e.* Combining this with the weak convergence above of $f_\varepsilon \rightharpoonup f$, we must have $f \equiv 1$ *a.e.*

In order to show that $\mathbb{F}_\varepsilon[u_\varepsilon; \mathbb{X}^n] \to \mathbb{F}[u; \mathbb{X}^n]$, we will need gamma convergence.

Definition 4.5.1 (Gamma-Convergence)**.** *Let (X, d) be a metric space and $F_k : X \to \mathbb{R}$ be a sequence of functionals on X. F_k is said to be Γ-convergent to the limit $F : X \to \mathbb{R}$ for $\mu \in X$ if and only if the following two conditions hold:*

1. *Limit Infimum Inequality: $\forall \mu \in X, \exists \{\mu_k\} \in X$ such that $\mu_k \to \mu$ as $k \to \infty$ and*

$$F(\mu) \leq \liminf_{k\to\infty} F_k(\mu_k). \tag{4.84}$$

2. *Limit Supremum Inequality: $\forall \mu \in X, \exists \{x_k\} \in X$ such that $\mu_k \to \mu$ as $k \to \infty$ and*

$$F(\mu) \geq \limsup_{k\to\infty} F_k(\mu_k). \tag{4.85}$$

We note that since u_ε is the minimiser of $\mathbb{F}$, this leads to the following set of inequalities:

$$\begin{aligned}0 \leq \mathbb{F}[u;\mathbb{X}^n] &= \int_{\mathbb{X}^n} F(|x|,|u|^2,|\nabla u|^2)dx \\ &\leq \liminf_{\varepsilon} \int_{\mathbb{X}^n} F(|x|,|u_\varepsilon|^2,|\nabla u_\varepsilon|^2)dx \\ &\leq \liminf_{\varepsilon} \int_{\mathbb{X}^n} F(|x|,|u_\varepsilon|^2,|\nabla u_\varepsilon|^2) + \frac{1}{\varepsilon}W(|u_\varepsilon|^2)dx \\ &\leq \limsup_{\varepsilon} \int_{\mathbb{X}^n} F(|x|,|u_\varepsilon|^2,|\nabla u_\varepsilon|^2) + \frac{1}{\varepsilon}W(|u_\varepsilon|^2)dx \\ &= \limsup_{\varepsilon} \mathbb{F}_\varepsilon[u_\varepsilon;\mathbb{X}^n] \leq \mathbb{F}[u;\mathbb{X}^n]. \end{aligned} \tag{4.86}$$

With equality throughout this shows $\mathbb{F}_\varepsilon[u_\varepsilon;\mathbb{X}^n] \to \mathbb{F}[u;\mathbb{X}^n]$.

Chapter 5

Generalisation of Spherical Twists Q to Spherical Whirls $\mathbf{Q}(\rho_1, \cdots, \rho_n)$

5.1 Introduction

In this section we consider maps $u \in \mathscr{C}(\overline{\mathbb{X}}^n, \mathbb{S}^{n-1})$ of the form

$$u : x \mapsto f(\rho_1, \dots, \rho_N)\mathbf{Q}(\rho_1, \dots, \rho_N)v(x), \quad x \in \overline{\mathbb{X}}^n, \tag{5.1}$$

where $\mathbf{Q} = \mathbf{Q}(\rho_1, \dots, \rho_N)$ is a continuous $\mathbf{SO}(n)$-valued map depending on the variable $x = (x_1, \dots, x_n)$ through $\rho = (\rho_1, \dots, \rho_N)$, that, depending on the dimension n being even or odd, we have the following description of ρ_j:

[**a**] (n even) writing $n = 2N$ we set $k = N$ and then

$$\rho_j = \sqrt{x_{2j-1}^2 + x_{2j}^2} \quad \text{for } 1 \leq j \leq N. \tag{5.2}$$

[**b**] (n odd) writing $n = 2N - 1$ we set $k = N - 1$ and then

$$\rho_j := \begin{cases} \sqrt{x_{2j-1}^2 + x_{2j}^2} & \text{for } 1 \leq j \leq N-1, \\ x_n & \text{for } j = N. \end{cases} \tag{5.3}$$

With this definition, we see that $\rho = (\rho_1, \dots, \rho_N)$ lies in the space

$$\mathbb{A}_N = \begin{cases} \{\rho \in \mathbb{R}_+^N : a \leq |\rho| \leq b\} & \text{for } n = 2N, \\ \{\rho \in \mathbb{R}_+^{N-1} \times \mathbb{R} : a \leq |\rho| \leq b\} & \text{for } n = 2N - 1. \end{cases} \tag{5.4}$$

These are known as spherical whirls.

Let $\mathsf{A} = \mathsf{A}(|u|^2, |\nabla u|^2)$ and $\mathsf{B} = \mathsf{B}(|u|^2, |\nabla u|^2)$ be two sufficiently regular real-valued functions. Consider the nonlinear system

$$
\begin{cases}
\mathscr{L}_C[u] = 0 & \text{in } \Omega, \\
|u| = 1 & \text{in } \Omega, \\
u = \varphi & \text{on } \partial\Omega,
\end{cases}
\tag{5.5}
$$

where the constrained differential operator $\mathscr{L}_C$ is given by

$$
\operatorname{div}[\mathsf{A}(|\nabla u|^2)\nabla u] + \mathsf{B}|\nabla u|^2 u = 0. \tag{5.6}
$$

5.2 The Constrained Case

In order to solve the constrained PDE

$$
\operatorname{div}[\mathsf{A}(|\nabla u|^2)\nabla u] + \mathsf{B}|\nabla u|^2 u = 0, \tag{5.7}
$$

we will need the following key identities.

Proposition 5.2.1 (Key identities for $u = \mathbf{Q}(\rho_1, \cdots, \rho_n)\theta$). *Let $u = \mathbf{Q}(\rho_1, \cdots, \rho_n)\theta$ be a spherical whirl on $\mathbb{X}^n$. Then the following identities hold for both the constrained case:*

(i) $\nabla u = \dfrac{1}{r}(\mathbf{Q} - \mathbf{Q}\theta \otimes \theta) + \displaystyle\sum_{l=1}^{n} \mathbf{Q}_{,l}\theta \otimes \nabla\rho_l,$

(ii) $|\nabla u|^2 = \dfrac{n-1}{r^2} + \displaystyle\sum_{l=1}^{n} |\mathbf{Q}_{,l}\theta|^2,$

(iii) $\Delta u = \displaystyle\sum_{l=1}^{n} \left[\mathbf{Q}_{,ll}\theta + \frac{2}{r}\mathbf{Q}_{,l}\nabla\rho_l + \left(\Delta\rho_l - \frac{2\rho_l}{r^2}\mathbf{Q}_{,l}\theta \right) \right] - \frac{n-1}{r^2}\mathbf{Q}\theta,$

and letting A denote the derivative in the third variable of $\mathsf{A}(r, s, \xi)$, and A_r, and A_ξ denote the derivatives with respect to the third and first variables, third and second variables, and a double derivative in the third variable respectively,

(iv) $\mathit{div}[\mathsf{A}(r, 1, |\nabla u|^2)\nabla u]$

$$
\begin{aligned}
&= \left(\mathsf{A}_r\theta + \mathsf{A}_s\left(2r^{-1}\theta(1 - \theta \otimes \theta)\right) + \mathsf{A}_\xi \left(- \left[\frac{2(n-1)\theta}{r^3} + \sum_{l=1}^{N} \nabla|\mathbf{Q}_{,l}\theta|^2 \right] \right) \right) \\
&\times \left(\frac{1}{r}(\mathbf{Q} - \mathbf{Q}\theta \otimes \theta) + \sum_{l=1}^{n} \mathbf{Q}_{,l}\theta \otimes \nabla\rho_l \right) \\
&+ \mathsf{A} \left(\sum_{l=1}^{n} \left[\mathbf{Q}_{,ll}\theta + \frac{2}{r}\mathbf{Q}_{,l}\nabla\rho_l + \left(\Delta\rho_l - \frac{2\rho_l}{r^2}\mathbf{Q}_{,l}\theta \right) \right] - \frac{n-1}{r^2}\mathbf{Q}\theta \right)
\end{aligned}
$$

Proof. Let $r = \sqrt{\rho_1^2 + \cdots + \rho_n^2}$. Taking the first derivative gives

$$\nabla u = \mathbf{Q}(\rho_1, \cdots, \rho_n)\nabla\theta + (\nabla\mathbf{Q}(\rho_1, \cdots, \rho_n))\theta \tag{5.8}$$

$$= \mathbf{Q}\nabla\theta + \sum_{l=1}^{n} \mathbf{Q}_{,l}\theta\nabla\rho_l, \tag{5.9}$$

with an application of the product and chain rule. An application of the identity $\nabla\theta = (\mathbf{I}_n - \theta\otimes\theta)r^{-1}$ gives the first identity. For the second identity, we apply the Hilbert-Schmidt norm to ∇u to get

$$\begin{aligned}
|\nabla u|^2 &= \mathrm{tr}[\nabla \mathrm{u}(\nabla \mathrm{u})^{\mathrm{t}}] \\
&= \mathrm{tr}\left\{\left[\frac{1}{\mathrm{r}}(\mathbf{Q} - \mathbf{Q}\theta\otimes\theta) + \sum_{\mathrm{l}=1}^{\mathrm{n}} \mathbf{Q}_{,\mathrm{l}}\theta\otimes\nabla\rho_{\mathrm{l}}\right]\left[\frac{1}{\mathrm{r}}(\mathbf{Q} - \mathbf{Q}\theta\otimes\theta) + \sum_{\mathrm{l}=1}^{\mathrm{n}} \mathbf{Q}_{,\mathrm{l}}\theta\otimes\nabla\rho_{\mathrm{l}}\right]^{\mathrm{t}}\right\} \\
&= \mathrm{tr}\left\{\frac{1}{\mathrm{r}^2}(\mathbf{I}_{\mathrm{n}} - \mathbf{Q}\theta\otimes\mathbf{Q}\theta) + \frac{1}{\mathrm{r}}\sum_{\mathrm{l}=1}^{\mathrm{n}}(\mathbf{Q} - \mathbf{Q}\theta\otimes\theta)(\nabla\rho_{\mathrm{l}}\otimes\mathbf{Q}_{,\mathrm{l}}\theta) + \sum_{\mathrm{l}=1}^{\mathrm{N}} \mathbf{Q}_{,\mathrm{l}}\theta\otimes\mathbf{Q}_{,\mathrm{l}}\theta\right\} \\
&= \frac{n-1}{r^2} + \sum_{l=1}^{n}\{\langle\mathbf{Q}\nabla\rho_l, \mathbf{Q}_{,l}\theta\rangle - \langle\mathbf{Q}\theta, \mathbf{Q}_{,l}\theta\rangle\langle\theta, \nabla\rho_l\rangle\} + r\sum_{l=1}^{n}|\mathbf{Q}_{,l}\theta|^2.
\end{aligned}$$

using $\nabla\rho_{,l}(\nabla\rho_{,l})^t = \mathbf{I}_n$. Using the fact that $\mathbf{Q}_{,l}^t\mathbf{Q}$ is skew symmetric, we can write

$$|\nabla u|^2 = \frac{n-1}{r^2} + \frac{1}{r}\sum_{l=1}^{n}\left\{\langle\mathbf{Q}\nabla\rho_l, \mathbf{Q}_{,l}\theta\rangle + r|\mathbf{Q}_{,l}\theta|^2\right\} \tag{5.10}$$

Proving that $\langle\mathbf{Q}\nabla\rho_l, \mathbf{Q}_{,l}\theta\rangle = 0$ will complete the proof of $|\nabla u|^2$. To this end, we know that

$$\langle\mathbf{Q}\nabla\rho_j, \mathbf{Q}_{,l}\theta\rangle = \langle\nabla\rho_j, \mathbf{Q}^t\mathbf{Q}_{,l}\theta\rangle \tag{5.11}$$

and $\theta = x/|x|$. Differentiating gives us

$$\begin{cases} \mathrm{diag}(\partial_l f_1\mathbf{J}, \cdots, \partial_l f_k\mathbf{J})|x|^{-1} & n = 2k, \\ \mathrm{diag}(\partial_l f_1\mathbf{J}y_1, \cdots, \partial_l f_k\mathbf{J}y_k, 0)|x|^{-1} & n = 2k+1. \end{cases} \tag{5.12}$$

Given that

$$\mathbf{Q}^t\mathbf{Q}_{,l}\frac{x}{|x|} = \begin{cases} (\partial_l f_1\mathbf{J}, \cdots, \partial_l f_k\mathbf{J})|x|^{-1} & n = 2k, \\ (\partial_l f_1\mathbf{J}y_1, \cdots, \partial_l f_k\mathbf{J}y_k, 0)|x|^{-1} & n = 2k+1, \end{cases} \tag{5.13}$$

this implies that

$$\frac{1}{|x|}\sum_{k=1}^{n}\frac{\partial_l f_j}{\rho_j}\langle y_j, \mathbf{J}y_j\rangle = 0 \tag{5.14}$$

for $n = 2k$ since the case $n = 2k+1$ is trivially true and utilizing the skew-symmetric nature of $\mathbf{J}$. This implies the second identity. For the third identity, to obtain Δu we take the divergence of ∇u to get the result. For the fourth identity, note that we can write

$$\mathscr{L}_C[u] = (\mathsf{A}_r\theta + \mathsf{A}_s\nabla u^2 + \nabla|\nabla u|^2\mathsf{A}_\xi)\nabla u + \mathsf{A}\Delta u.$$

Observe that the gradient of $|\nabla u|^2$ is given by

$$\nabla|\nabla u|^2 = -\left[\frac{2(n-1)\theta}{r^3} + \sum_{l=1}^{N} \nabla|\mathbf{Q}_{,l}\theta|^2\right]. \tag{5.15}$$

Substituting the identities for ∇u, $\nabla|\nabla u|^2$ and Δu and using $\nabla u^2 = \nabla[\mathbf{Q}\theta] = 2r^{-1}\theta(1 - \theta \otimes \theta)$ gives us the result giving the fourth identity. □

Substituting the identities from proposition (5.2.1) into the differential operator

$$\operatorname{div}[\mathsf{A}(|\nabla u|^2)\nabla u] + \mathsf{B}|\nabla u|^2 u = 0 \tag{5.16}$$

allows us to rewrite this identity as

$$\operatorname{div}\left[\mathsf{A}\left(\frac{n-1}{r^2} + \sum_{l=1}^{n} |\mathbf{Q}_{,l}\theta|^2\right)\left(\frac{1}{r}(\mathbf{Q} - \mathbf{Q}\theta \otimes \theta) + \sum_{l=1}^{n} \mathbf{Q}_{,l}\theta \otimes \nabla\rho_l\right)\right]$$
$$+ \mathsf{B}\left(\frac{n-1}{r^2} + \sum_{l=1}^{n} |\mathbf{Q}_{,l}\theta|^2\right)\mathbf{Q}\theta = 0.$$

First we will verify that the expanded differential operator

$$\mathsf{A}(|\nabla u|^2)\Delta u + \nabla u \nabla \mathsf{A}(|\nabla u|^2) + \mathsf{B}|\nabla u|^2 u = 0. \tag{5.17}$$

holds using the following identities.

Theorem 5.2.1 (Key identities for $u = \mathbf{Q}\theta$)**.** *Let $r = |x|$ and $\theta = x|x|^{-1}$. Suppose $u = \mathbf{Q}(r)\theta$ is a spherical twist defined by a twice continuously differentiable twist path $\mathbf{Q}$. Then the following identities hold for both the constrained and unconstrained cases:*

(i) $\nabla u = r^{-1}\left(\mathbf{Q} + (r\dot{\mathbf{Q}} - \mathbf{Q})\theta \otimes \theta\right)$,

(ii) $|\nabla u|^2 = \dfrac{n-1}{r^2} + |\dot{\mathbf{Q}}\theta|^2$,

(iii) $\Delta u = \left((n-1)(r\dot{\mathbf{Q}} - \mathbf{Q}) + r^2\ddot{\mathbf{Q}}\right)\theta r^{-2}$,

(iv) $\nabla|\nabla u|^2 = 2\left[\left\langle \ddot{\mathbf{Q}}\theta, \dot{\mathbf{Q}}\theta\right\rangle - \dfrac{|\dot{\mathbf{Q}}\theta|^2}{r} - \dfrac{n-1}{r^3} + \dfrac{\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}}{r}\right]\theta$,

Proof. For a proof of identity $i), ii)$ and $iii)$, see Proposition 5.2.1. To prove identity $iv)$, let $\dot{\mathbf{Q}} = A$. Then $\nabla|\dot{\mathbf{Q}}\theta|^2$ can be rewritten as

$$\nabla|A\theta|^2 = \nabla\langle Ax/|x|, Ax/|x|\rangle = \nabla\left\{\frac{1}{r^2}\langle A^t Ax, x\rangle\right\}. \tag{5.18}$$

Let $F = A^t A$. Then

$$\nabla\left\{\frac{1}{r^2}\langle Fx, x\rangle\right\} = \frac{-2\theta}{r^3}\langle Fx, x\rangle + \frac{1}{r^2}\nabla\langle Fx, x\rangle. \tag{5.19}$$

Since $F = A^tA$, this implies that $\dot{F} = \dot{A}^tA + A^t\dot{A}$. Expanding $\nabla\langle Fx, x\rangle$ gives us

$$\nabla\langle Fx, x\rangle = 2Fx + \langle \dot{F}x, x\rangle\theta = 2Fx + \langle(\dot{A}^tA + A^t\dot{A})x, x\rangle\theta. \tag{5.20}$$

This can be rewritten as

$$\nabla\langle Fx, x\rangle = 2A^tAx + 2\langle \dot{A}x, Ax\rangle\theta \tag{5.21}$$

Using the identity (5.21) and substituting back into (5.19) gives us

$$\begin{aligned}\nabla\left\{\frac{1}{r^2}\langle Fx, x\rangle\right\} &= -\frac{2}{r^3}|Ax|^2\theta + \frac{1}{r^2}\left\{2A^tAx + 2\langle \dot{A}x, Ax\rangle\theta\right\} \\ &= -2\frac{|A\theta|^2}{r}\theta + 2\langle \dot{A}\theta, A\theta\rangle\theta + \frac{2}{r}A^tA\theta. \end{aligned}\tag{5.22}$$

This implies that

$$\nabla|\nabla u|^2 = 2\left[\left\langle \ddot{\mathbf{Q}}\theta, \dot{\mathbf{Q}}\theta\right\rangle - \frac{|\dot{\mathbf{Q}}\theta|^2}{r} - \frac{n-1}{r^3} + \frac{\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}}{r}\right]\theta \tag{5.23}$$

which is the result we want. □

Given these identities, we can now proceed with the following theorem.

Theorem 5.2.2. *Let $r = |x|$ and $\theta = x|x|^{-1}$. Suppose $u = \mathbf{Q}(r)\theta$ is a spherical twist defined by a twice continuously differentiable twist path $\mathbf{Q}$. Then the following identity holds*

$$\begin{aligned}\mathscr{L}_C[u] &= \operatorname{div}\left[\mathsf{A}(|\nabla u|^2)\nabla u\right] + \mathsf{B}|\nabla u|^2u \\ &= \frac{2\mathsf{A}_\xi}{r}\left(\left\langle \ddot{\mathbf{Q}}\theta, \dot{\mathbf{Q}}\theta\right\rangle \mathbf{Q} - \frac{|\dot{\mathbf{Q}}\theta|^2\mathbf{Q}}{r} - \frac{(n-1)\mathbf{Q}}{r^3} + \frac{\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\mathbf{Q}}{r}\right)\theta \\ &+ \frac{2\mathsf{A}_\xi}{r}\left((r\dot{\mathbf{Q}} - \mathbf{Q})\left\langle \ddot{\mathbf{Q}}\theta, \dot{\mathbf{Q}}\theta\right\rangle - (r\dot{\mathbf{Q}} - \mathbf{Q})\frac{|\dot{\mathbf{Q}}\theta|^2}{r}\right)\theta \\ &+ \frac{2\mathsf{A}_\xi}{r}\left(-(r\dot{\mathbf{Q}} - \mathbf{Q})\frac{n-1}{r^3} + (r\dot{\mathbf{Q}} - \mathbf{Q})\frac{\dot{\mathbf{Q}}^t\mathbf{Q}}{r}\right)\theta \\ &+ \frac{\mathsf{A}}{r^2}\left[\left((n-1)(r\dot{\mathbf{Q}} - \mathbf{Q}) + r^2\ddot{\mathbf{Q}}\right)\theta\right] + \mathsf{B}\left(\frac{n-1}{r^2} + |\dot{\mathbf{Q}}\theta|^2\right)\mathbf{Q}\theta. \end{aligned}\tag{5.24}$$

Proof. To prove this identity, an application of the chain and product rule give us the identity

$$\mathscr{L}_C[u] = \mathsf{A}_\xi\nabla u[\nabla|\nabla u|^2] + \mathsf{A}\Delta u + \mathsf{B}|\nabla u|^2u. \tag{5.25}$$

Substituting the appropriate identities from Theorem 5.2.1 into (5.25) gives us the result

$$\begin{aligned}\mathscr{L}_C[u] = &\frac{2\mathsf{A}_\xi}{r}\left[\mathbf{Q} + (r\dot{\mathbf{Q}} - \mathbf{Q})\theta\otimes\theta\right]\left[\left\langle \ddot{\mathbf{Q}}\theta, \dot{\mathbf{Q}}\theta\right\rangle - \frac{|\dot{\mathbf{Q}}\theta|^2}{r} - \frac{n-1}{r^3} + \frac{\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}}{r}\right]\theta \\ &+ \frac{\mathsf{A}}{r^2}\left[\left((n-1)(r\dot{\mathbf{Q}} - \mathbf{Q}) + r^2\ddot{\mathbf{Q}}\right)\theta\right] + \mathsf{B}\left(\frac{n-1}{r^2} + |\dot{\mathbf{Q}}\theta|^2\right)\mathbf{Q}\theta.\end{aligned}$$

Expanding the double brackets above gives us

$$
\begin{aligned}
\mathscr{L}_C[u] = {} & \frac{2\mathsf{A}_\xi}{r}\left(\left\langle \ddot{\mathbf{Q}}\theta, \dot{\mathbf{Q}}\theta \right\rangle \mathbf{Q} - \frac{|\dot{\mathbf{Q}}\theta|^2\mathbf{Q}}{r} - \frac{(n-1)\mathbf{Q}}{r^3} + \frac{\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}\mathbf{Q}}{r}\right)\theta \\
& + \frac{2\mathsf{A}_\xi}{r}\left((r\dot{\mathbf{Q}} - \mathbf{Q})\left\langle \ddot{\mathbf{Q}}\theta, \dot{\mathbf{Q}}\theta \right\rangle - (r\dot{\mathbf{Q}} - \mathbf{Q})\frac{|\dot{\mathbf{Q}}\theta|^2}{r}\right)\theta \\
& + \frac{2\mathsf{A}_\xi}{r}\left(-(r\dot{\mathbf{Q}} - \mathbf{Q})\frac{n-1}{r^3} + (r\dot{\mathbf{Q}} - \mathbf{Q})\frac{\dot{\mathbf{Q}}^t\dot{\mathbf{Q}}}{r}\right)\theta \\
& + \frac{\mathsf{A}}{r^2}\left[\left((n-1)(r\dot{\mathbf{Q}} - \mathbf{Q}) + r^2\ddot{\mathbf{Q}}\right)\theta\right] + \mathsf{B}\left(\frac{n-1}{r^2} + |\dot{\mathbf{Q}}\theta|^2\right)\mathbf{Q}\theta, \qquad (5.26)
\end{aligned}
$$

which is the result we want. □

Further simplification gives us the expression

$$
\begin{aligned}
\mathscr{L}_C[u] & = \operatorname{div}\left[\mathsf{A}(|\nabla u|^2)\nabla u\right] + \mathsf{B}|\nabla u|^2 u \\
& = \frac{2\mathsf{A}_\xi}{r}\left(\left\langle \ddot{\mathbf{Q}}\theta, \dot{\mathbf{Q}}\theta \right\rangle r\dot{\mathbf{Q}} - |\dot{\mathbf{Q}}\theta|^2\dot{\mathbf{Q}} - \frac{(n-1)\dot{\mathbf{Q}}}{r^2} + \dot{\mathbf{Q}}^t|\dot{\mathbf{Q}}|^2\right)\theta \\
& + \frac{\mathsf{A}}{r^2}\left[\left((n-1)(r\dot{\mathbf{Q}} - \mathbf{Q}) + r^2\ddot{\mathbf{Q}}\right)\theta\right] + \mathsf{B}\left(\frac{n-1}{r^2} + |\dot{\mathbf{Q}}\theta|^2\right)\mathbf{Q}\theta. \qquad (5.27)
\end{aligned}
$$

Appendix A

Gegenbauer Polynomials for $1 \leq m \leq 10$

We can obtain the Gegenbauer polynomials for $1 \leq m \leq 10$ by substituting $\alpha = \nu - 1/2$ and $\beta = \nu - 1/2$ into the Jacobi polynomials. The first three Gegenbauer polynomials are given by

$$\mathscr{G}_1(\nu) = \frac{-X}{2\nu+1}, \ \mathscr{G}_2(\nu) = \frac{3X^2 - 4\nu X}{(2\nu+1)(2\nu+3)}, \ \mathscr{G}_3(\nu) = \frac{-15X^3 + 60\nu X^2 - (64\nu^2 + 16\nu)X}{(2\nu+1)(2\nu+3)(2\nu+5)}.$$

For $m = 4$ we can write

$$\mathscr{G}_4(\nu) = \frac{(-1)^j \sum_{j=1}^{4} \mathsf{Q}_j^4 X^j}{\mathscr{A}_4(\nu)} \tag{A.1}$$

where $\mathscr{A}_4(\nu) = (2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)$ and

$$\mathsf{Q}_1^4(\nu) = 2176\nu^3 + 1536\nu^2 + 320\nu, \mathsf{Q}_2^4(\nu) = 2352\nu^2 + 672\nu, \ \mathsf{Q}_3^4(\nu) = 840\nu,$$
$$\mathsf{Q}_4^4(\nu) = 105.$$

For $m = 5$ we can write

$$\mathscr{G}_5(\nu) = \frac{(-1)^j \sum_{j=1}^{5} \mathsf{Q}_j^5 X^j}{\mathscr{A}_5(\nu)} \tag{A.2}$$

where $\mathscr{A}_5(\nu) = (2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)$ and

$$\mathsf{Q}_1^5(\nu) = 126976\nu^4 + 172032\nu^3 + 88064\nu^2 + 16128\nu$$
$$\mathsf{Q}_2^5(\nu) = 151680\nu^3 + 120960\nu^2 + 29760\nu, \ \mathsf{Q}_3^5(\nu) = 65520\nu^2 + 20160\nu,$$
$$\mathsf{Q}_4^5(\nu) = 12600\nu, \ \mathsf{Q}_5^5(\nu) = 945.$$

For $m = 6$ we can write

$$\mathscr{G}_6(\nu) = \frac{(-1)^j \sum_{j=1}^{6} \mathsf{Q}_j^6 X^j}{\mathscr{A}_6(\nu)} \tag{A.3}$$

where

$$\mathscr{A}_6(\nu) = (2\nu+1)(2\nu+3)(2\nu+5)(2\nu+7)(2\nu+9)(2\nu+11) \tag{A.4}$$

and

$$\begin{aligned}
\mathsf{Q}_1^6(\nu) &= 11321344\nu^5 + 24838144\nu^4 + 22575104\nu^3 + 9703424\nu^2 + 1612800\nu \\
\mathsf{Q}_2^6(\nu) &= 14581248\nu^4 + 22099968\nu^3 + 13018368\nu^2 + 2838528\nu \\
\mathsf{Q}_3^6(\nu) &= 7149120\nu^3 + 6145920\nu^2 + 1657920\nu,\ \mathsf{Q}_4^6(\nu) = 1718640\nu^2 + 554400\nu, \\
\mathsf{Q}_5^6(\nu) &= 207900\nu,\ \mathsf{Q}_6^6(\nu) = 10395.
\end{aligned}$$

For $m = 7$ we can write

$$\mathscr{G}_7(\nu) = \frac{(-1)^j \sum_{j=1}^{7} \mathsf{Q}_j^7 X^j}{\mathscr{A}_7(\nu)} \tag{A.5}$$

where $\mathscr{A}_7(\nu) = (2\nu+1)(2\nu+3)\cdots(2\nu+11)(2\nu+13)$ and

$$\begin{aligned}
\mathsf{Q}_1^7(\nu) &= 1431568384\nu^6 + 4613308416\nu^5 + 6456770560\nu^4 + 4771676160\nu^3 \\
&\quad + 1819426816\nu^2 + 280203264\nu, \\
\mathsf{Q}_2^7(\nu) &= 1956864000\nu^5 + 4760770560\nu^4 + 4894955520\nu^3 + 2434897920\nu^2 \\
&\quad + 482227200\nu, \\
\mathsf{Q}_3^7(\nu) &= 1053212160\nu^4 + 1718196480\nu^3 + 1103182080\nu^2 + 265224960\nu, \\
\mathsf{Q}_4^7(\nu) &= 293092800\nu^3 + 265224960\nu^2 + 75915840\nu,\ \mathsf{Q}_5^7(\nu) = 45405360\nu^2 + 15135120\nu, \\
\mathsf{Q}_6^7(\nu) &= 3783780\nu,\ \mathsf{Q}_7^7(\nu) = 135135.
\end{aligned}$$

For $m = 8$ we can write

$$\mathscr{G}_8(\nu) = \frac{(-1)^j \sum_{j=1}^{8} \mathsf{Q}_j^8 X^j}{\mathscr{A}_8(\nu)} \tag{A.6}$$

where $\mathscr{A}_8(\nu) = (2\nu+1)(2\nu+3)\cdots(2\nu+13)(2\nu+15)$ and

$$\begin{aligned}
\mathsf{Q}_1^8(\nu) &= 243680935936\nu^7 + 1082098974720\nu^6 + 2147343400960\nu^5 \\
&\quad + 2385562828800\nu^4 + 1536703627264\nu^3 + 535343923200\nu^2 \\
&\quad + 77491814400\nu, \\
\mathsf{Q}_2^8(\nu) &= 349753884672\nu^6 + 1240160993280\nu^5 + 1938656870400\nu^4 \\
&\quad + 1627076689920\nu^3 + 718825156608\nu^2 + 131695534080\nu,
\end{aligned}$$

$$\mathsf{Q}_3^8(\nu) = 202468331520\nu^5 + 528789381120\nu^4 + 589432596480\nu^3 + 320853012480\nu^2 + 70195507200\nu,$$

$$\mathsf{Q}_4^8(\nu) = 62696874240\nu^4 + 107796648960\nu^3 + 73432719360\nu^2 + 18819440640\nu,$$

$$\mathsf{Q}_5^8(\nu) = 11416204800\nu^3 + 10724313600\nu^2 + 3199996800\nu,$$

$$\mathsf{Q}_6^8(\nu) = 1241079840\nu^2 + 423783360\nu,\ \mathsf{Q}_7^8(\nu) = 75675600\nu,\ \mathsf{Q}_8^8(\nu) = 2027025.$$

For $m = 9$ we can write

$$\mathscr{G}_9(\nu) = \frac{(-1)^j \sum_{j=1}^{9} \mathsf{Q}_j^9 X^j}{\mathscr{A}_9(\nu)} \tag{A.7}$$

where $\mathscr{A}_9(\nu) = (2\nu+1)(2\nu+3)\cdots(2\nu+15)(\nu+17)$ and

$$\mathsf{Q}_1^9(\nu) = 53725527801856\nu^8 + 314195228753920\nu^7 + 836893822418944\nu^6 + 1293678605762560\nu^5 + 1235127038377984\nu^4 + 718747992064000\nu^3 + 233041183113216\nu^2 + 32032902021120\nu,$$

$$\mathsf{Q}_2^9(\nu) = 80346749337600\nu^7 + 389903414722560\nu^6 + 855492223303680\nu^5 + 1064413574922240\nu^4 + 779472386949120\nu^3 + 314441581363200\nu^2 + 54011794636800\nu,$$

$$\mathsf{Q}_3^9(\nu) = 49350347243520\nu^6 + 187277662126080\nu^5 + 315825379491840\nu^4 + 288185441034240\nu^3 + 139522688102400\nu^2 + 28247291043840\nu,$$

$$\mathsf{Q}_4^9(\nu) = 16605304934400\nu^5 + 45695027589120\nu^4 + 53974013184000\nu^3 + 31276484628480\nu^2 + 7309641830400\nu,$$

$$\mathsf{Q}_5^9(\nu) = 3401025788160\nu^4 + 6083384186880\nu^3 + 4328471347200\nu^2 + 1160924244480\nu,$$

$$\mathsf{Q}_6^9(\nu) = 439904424960\nu^3 + 425201736960\nu^2 + 130853923200\nu,$$

$$\mathsf{Q}_7^9(\nu) = 35506991520\nu^2 + 12350257920\nu,\ \mathsf{Q}_8^9(\nu) = 1654052400\nu,$$

$$\mathsf{Q}_9^9(\nu) = 34459425.$$

For $m = 10$ we can write

$$\mathscr{G}_{10}(\nu) = \frac{(-1)^j \sum_{j=1}^{10} \mathsf{Q}_j^{10} X^j}{\mathscr{A}_{10}(\nu)} \tag{A.8}$$

where $\mathscr{A}_{10}(\nu) = (2\nu+1)(2\nu+3)\cdots(2\nu+17)(\nu+19)$ and

$$
\begin{aligned}
\mathsf{Q}_1^{10}(\nu) &= 14893509177769984\nu^9 + 110883708088614912\nu^8 \\
&\quad + 381139340123701248\nu^7 + 778957062741688320\nu^6 \\
&\quad + 1022782830066794496\nu^5 + 873812349595680768\nu^4 \\
&\quad + 469441189531615232\nu^3 + 143464484343644160\nu^2 \\
&\quad + 18869769201254400\nu, \\
\mathsf{Q}_2^{10}(\nu) &= 23074296051007488\nu^8 + 146583040875823104\nu^7 \\
&\quad + 428130363788230656\nu^6 + 733176962875392000\nu^5 \\
&\quad + 784593411349020672\nu^4 + 518911625966125056\nu^3 \\
&\quad + 194564974050607104\nu^2 + 31648507196866560\nu, \\
\mathsf{Q}_3^{10}(\nu) &= 14893024956579840\nu^7 + 77107626344448000\nu^6 \\
&\quad + 181695753079357440\nu^5 + 244389707466670080\nu^4 \\
&\quad + 194786177757265920\nu^3 + 86136301334200320\nu^2 \\
&\quad + 16344496001433600\nu, \\
\mathsf{Q}_4^{10}(\nu) &= 5360011008737280\nu^6 + 21410590542888960\nu^5 + 38191786546421760\nu^4 \\
&\quad + 37019397146296320\nu^3 + 19109065550991360\nu^2 + 4137013456773120\nu, \\
\mathsf{Q}_5^{10}(\nu) &= 1202713026278400\nu^5 + 3445912292659200\nu^4 + 4253165783193600\nu^3 \\
&\quad + 2581732132761600\nu^2 + 632775890073600\nu, \\
\mathsf{Q}_6^{10}(\nu) &= 176538703461120\nu^4 + 325701001866240\nu^3 + 239638523604480\nu^2 \\
&\quad + 66507903221760\nu, \\
\mathsf{Q}_7^{10}(\nu) &= 17124220713600\nu^3 + 16928674963200\nu^2 + 5335605475200\nu, \\
\mathsf{Q}_8^{10}(\nu) &= 1068517850400\nu^2 + 377123947200\nu, \\
\mathsf{Q}_9^{10}(\nu) &= 39283744500\nu, \ \ \mathsf{Q}_{10}^{10}(\nu) = 654729075.
\end{aligned}
$$

Appendix B

Jacobi Polynomials for $1 \leq m \leq 10$

By making the substitutions $a = -k$, $b = k + \alpha + \beta + 1$ and $c = \alpha + 1$, we can transform the hypergeometric series into Jacobi polynomials. The first three Jacobi polynomials are given by

$$\mathscr{R}_1(X) = \frac{-X}{2(\alpha+1)},\ \mathscr{R}_2(X) = \frac{3X^2 - (\alpha+3\beta+2)X}{4(\alpha+1)(\alpha+2)}, \tag{B.1}$$

$$\mathscr{R}_3(X) = \frac{-15X^3 + 15(\alpha+3\beta+2)X^2 - (4\alpha^2+30\alpha\beta+30\beta^2+20\alpha+60\beta+24)X}{8(\alpha+1)(\alpha+2)(\alpha+3)} \tag{B.2}$$

For $m = 4$ we can write

$$\mathscr{R}_4(X) = \sum_{j=1}^{4} (-1)^j \frac{Q_j^4(\alpha,\beta)}{2^4 \mathscr{A}_4(\alpha)} X^j \tag{B.3}$$

where $\mathscr{A}_4(\alpha) = (\alpha+1)(\alpha+2)(\alpha+3)(\alpha+4)$ and

$$\begin{aligned} Q_1^4(\alpha,\beta) &= 34\alpha^3 + 462\alpha^2\beta + 1050\alpha\beta^2 + 630\beta^3 + 306\alpha^2 + 2184\alpha\beta + 2310\beta^2 + 884\alpha \\ &\quad + 2604\beta + 816, \\ Q_2^4(\alpha,\beta) &= 147\alpha^2 + 1050\alpha\beta + 1155\beta^2 + 714\alpha + 2310\beta + 924, \\ Q_3^4(\alpha,\beta) &= 210\alpha + 630\beta + 420,\ Q_4^4(\alpha,\beta) = 105. \end{aligned}$$

For $m = 5$ we can write

$$\mathscr{R}_5(X) = \sum_{j=1}^{5} (-1)^j \frac{Q_j^5(\alpha,\beta)}{2^5 \mathscr{A}_5(\alpha)} X^j \tag{B.4}$$

where $\mathscr{A}_5(\alpha) = (\alpha+1)(\alpha+2)\cdots(\alpha+5)$ and

$$\begin{aligned} Q_1^5(\alpha,\beta) &= 496\alpha^4 + 10560\alpha^3\beta + 40320\alpha^2\beta^2 + 52920\alpha\beta^3 + 22680\beta^4 + 6944\alpha^3 + 86400\alpha^2\beta \\ &\quad + 204120\alpha\beta^2 + 128520\beta^3 + 35216\alpha^2 + 239280\alpha\beta + 262080\beta^2 + 76384\alpha \\ &\quad + 220320\beta + 59520, \end{aligned} \tag{B.5}$$

$$\begin{aligned} Q_2^5(\alpha,\beta) &= 2370\alpha^3 + 29610\alpha^2\beta + 72450\alpha\beta^2 + 47250\beta^3 + 20250\alpha^2 + 151200\alpha\beta + 177030\beta^2 \\ &\quad + 61620\alpha + 202860\beta + 64080, \end{aligned} \tag{B.6}$$

$$Q_3^5(\alpha,\beta) = 4095\alpha^2 + 28350\alpha\beta + 33075\beta^2 + 19530\alpha + 66150\beta + 26460, \tag{B.7}$$

$$Q_4^5(\alpha,\beta) = 3150\alpha + 9450\beta + 6300,\ \ Q_5^5(\alpha,\beta) = 945. \tag{B.8}$$

For $m = 6$ we can write

$$\mathscr{R}_6(X) = \sum_{j=1}^{6} (-1)^j \frac{Q_j^6(\alpha,\beta)}{2^6 \mathscr{A}_6(\alpha)} X^j \tag{B.9}$$

where $\mathscr{A}_6(\alpha) = (\alpha+1)(\alpha+2)\cdots(\alpha+6)$ and

$$\begin{aligned} Q_1^6(\alpha,\beta) &= 11056\alpha^5 + 338448\alpha^4\beta + 1907400\alpha^3\beta^2 + 4074840\alpha^2\beta^3 + 3742200\alpha\beta^4 \\ &\quad + 1247400\beta^5 + 221120\alpha^4 + 4184664\alpha^3\beta + 16386480\alpha^2\beta^2 + 22370040\alpha\beta^3 \\ &\quad + 9979200\beta^4 + 1713680\alpha^3 + 19813992\alpha^2\beta + 47842080\alpha\beta^2 + 31185000\beta^3 \\ &\quad + 6412480\alpha^2 + 41794896\alpha\beta + 46767600\beta^2 + 11542464\alpha + 32662080\beta \\ &\quad + 7960320, \end{aligned} \tag{B.10}$$

$$\begin{aligned} Q_2^6(\alpha,\beta) &= 56958\alpha^4 + 1087020\alpha^3\beta + 4393620\alpha^2\beta^2 + 6195420\alpha\beta^3 + 2848230\beta^4 \\ &\quad + 743754\alpha^3 + 9465390\alpha^2\beta + 24358950\alpha\beta^2 + 16694370\beta^3 + 3900072\alpha^2 \\ &\quad + 28963440\alpha\beta + 35176680\beta^2 + 9321576\alpha + 30400920\beta + 8347680, \end{aligned} \tag{B.11}$$

$$\begin{aligned} Q_3^6(\alpha,\beta) &= 111705\alpha^3 + 1320165\alpha^2\beta + 3378375\alpha\beta^2 + 2338875\beta^3 + 923670\alpha^2 \\ &\quad + 7068600\alpha\beta + 8877330\beta^2 + 2895420\alpha + 10270260\beta + 3259080, \end{aligned} \tag{B.12}$$

$$Q_4^6(\alpha,\beta) = 107415\alpha^2 + 727650\alpha\beta + 883575\beta^2 + 505890\alpha + 1767150\beta + 706860, \tag{B.13}$$

$$Q_5^6(\alpha,\beta) = 51975\alpha + 155925\beta + 103950, \tag{B.14}$$

$$Q_6^6(\alpha,\beta) = 10395. \tag{B.15}$$

B.0.1 Jacobi Polynomials Cases $m = 7$ and 8

For $m = 7$ we can write

$$\mathscr{R}_7(X) = \sum_{j=1}^{7} (-1)^j \frac{Q_j^7(\alpha,\beta)}{2^7 \mathscr{A}_7(\alpha)} X^j \tag{B.16}$$

where $\mathscr{A}_7(\alpha) = (\alpha+1)(\alpha+2)\cdots(\alpha+7)$ and

$$\begin{aligned}
Q_1^7(\alpha,\beta) &= 349504\alpha^6 + 14523600\alpha^5\beta + 112192080\alpha^4\beta^2 + 344504160\alpha^3\beta^3 \\
&\quad + 505945440\alpha^2\beta^4 + 356756400\alpha\beta^5 + 97297200\beta^6 + 9436608\alpha^5 \\
&\quad + 250897920\alpha^4\beta + 1434713280\alpha^3\beta^2 + 3165402240\alpha^2\beta^3 \\
&\quad + 3009726720\alpha\beta^4 + 1037836800\beta^5 + 103103680\alpha^4 \\
&\quad + 1779064560\alpha^3\beta + 7044557520\alpha^2\beta^2 + 9884674800\alpha\beta^3 \\
&\quad + 4547022480\beta^4 + 581924160\alpha^3 + 6349324800\alpha^2\beta \\
&\quad + 15505089600\alpha\beta^2 + 10361070720\beta^3 + 1783868416\alpha^2 \\
&\quad + 11241746880\alpha\beta + 12745212480\beta^2 + 2805818112\alpha \\
&\quad + 7811032320\beta + 1761500160, \\
Q_2^7(\alpha,\beta) &= 1911000\alpha^5 + 51471420\alpha^4\beta + 302942640\alpha^3\beta^2 + 688647960\alpha^2\beta^3 \\
&\quad + 673512840\alpha\beta^4 + 238378140\beta^5 + 35162400\alpha^4 + 669128460\alpha^3\beta \\
&\quad + 2816753940\alpha^2\beta^2 + 4146482340\alpha\beta^3 + 1985403420\beta^4 + 277684680\alpha^3 \\
&\quad + 3448525080\alpha^2\beta + 9116507400\alpha\beta^2 + 6465939480\beta^3 + 1125065760\alpha^2 \\
&\quad + 8145817680\alpha\beta + 10093683600\beta^2 + 2279353440\alpha + 7300893600\beta \\
&\quad + 1820407680, \\
Q_3^7(\alpha,\beta) &= +4114110\alpha^4 + 72882810\alpha^3\beta + 304594290\alpha^2\beta^2 + 452161710\alpha\beta^3 \\
&\quad + 219459240\beta^4 + 51381330\alpha^3 + 659188530\alpha^2\beta + 1799187390\alpha\beta^2 \\
&\quad + 1314863550\beta^3 + 273993720\alpha^2 + 2158196040\alpha\beta + 2828105280\beta^2 \\
&\quad + 698257560\alpha + 2485402920\beta + 689008320, \\
Q_4^7(\alpha,\beta) &= 4579575\alpha^3 + 52026975\alpha^2\beta + 137162025\alpha\beta^2 + 99324225\beta^3 \\
&\quad + 37026990\alpha^2 + 287567280\alpha\beta + 380269890\beta^2 + 118258140\alpha \\
&\quad + 442702260\beta + 140900760, \\
Q_5^7(\alpha,\beta) &= 2837835\alpha^2 + 18918900\alpha\beta + 23648625\beta^2 + 13243230\alpha \\
&\quad + 47297250\beta + 18918900, \\
Q_6^7(\alpha,\beta) &= 945945\alpha + 1891890 + 2837835\beta,\ Q_7^7(\alpha,\beta) = 135135.
\end{aligned}$$

For $m = 8$ we can write

$$\mathscr{R}_8(X) = \sum_{j=1}^{8} (-1)^j \frac{Q_j^8(\alpha,\beta)}{2^8 \mathscr{A}_8(\alpha)} X^j \tag{B.17}$$

where $\mathscr{A}_8(\alpha) = (\alpha+1)(\alpha+2)\cdots(\alpha+8)$ and

$$
\begin{aligned}
Q_1^8(\alpha,\beta) &= 14873104\alpha^7 + 804913392\alpha^6\beta + 8117600400\alpha^5\beta^2 \\
&\quad + 33403209840\alpha^4\beta^3 + 69437768400\alpha^3\beta^4 + 77416138800\alpha^2\beta^5 \\
&\quad + 44270226000\alpha\beta^6 + 10216206000\beta^7 + 520558640\alpha^6 \\
&\quad + 18444541536\alpha^5\beta + 143318382480\alpha^4\beta^2 + 451856845440\alpha^3\beta^3 \\
&\quad + 684031748400\alpha^2\beta^4 + 497188692000\alpha\beta^5 + 139621482000\beta^6 \\
&\quad + 7600156144\alpha^5 + 181464867120\alpha^4\beta + 1040113226880\alpha^3\beta^2 \\
&\quad + 2344611709440\alpha^2\beta^3 + 2288462576400\alpha\beta^4 + 810712702800\beta^5 \\
&\quad + 59864243600\alpha^4 + 961995643200\alpha^3\beta + 3820133378880\alpha^2\beta^2 \\
&\quad + 5463008591040\alpha\beta^3 + 2572894724400\beta^4 + 274022068096\alpha^3 \\
&\quad + 2856724390848\alpha^2\beta + 7012545583680\alpha\beta^2 + 4771195228800\beta^3 \\
&\quad + 726699861440\alpha^2 + 4456757931264\alpha\beta + 5093848468800\beta^2 \\
&\quad + 1030170675456\alpha + 2828565169920\beta + 599683553280, \\
Q_2^8(\alpha,\beta) &= 85389132\alpha^6 + 3078752040\alpha^5\beta + 24563038500\alpha^4\beta^2 \\
&\quad + 79608569040\alpha^3\beta^3 + 123739876260\alpha^2\beta^4 + 92189097000\alpha\beta^5 \\
&\quad + 26489162700\beta^6 + 2098317096\alpha^5 + 55362598200\alpha^4\beta \\
&\quad + 336603627360\alpha^3\beta^2 + 794407854240\alpha^2\beta^3 + 805799194200\alpha\beta^4 \\
&\quad + 295151056200\beta^5 + 23090987820\alpha^4 + 422041969440\alpha^3\beta \\
&\quad + 1810067018760\alpha^2\beta^2 + 2742081582240\alpha\beta^3 + 1353354862860\beta^4 \\
&\quad + 139135401360\alpha^3 + 1662176028240\alpha^2\beta + 4448249966160\alpha\beta^2 \\
&\quad + 3229616229840\beta^3 + 472078323408\alpha^2 + 3318482512800\alpha\beta \\
&\quad + 4154960169360\beta^2 + 843310275264\alpha + 2649975531840\beta \\
&\quad + 614324632320, \\
Q_3^8(\alpha,\beta) &= 197722980\alpha^5 + 4870805940\alpha^4\beta + 29362132800\alpha^3\beta^2 \\
&\quad + 69772903200\alpha^2\beta^3 + 71645874300\alpha\beta^4 + 26618892300\beta^5 \\
&\quad + 3446106300\alpha^4 + 65250745560\alpha^3\beta + 288574846560\alpha^2\beta^2 \\
&\quad + 449788739400\alpha\beta^3 + 227899772100\beta^4 + 27387556560\alpha^3 \\
&\quad + 356228832960\alpha^2\beta + 1006669263600\alpha\beta^2 + 762896534400\beta^3 \\
&\quad + 116890824960\alpha^2 + 912271520160\alpha\beta + 1222109048160\beta^2 \\
&\quad + 257347752480\alpha + 903441258720\beta + 228366552960,
\end{aligned}
$$

$$\begin{aligned}
Q_4^8(\alpha,\beta) &= 244909665\alpha^4 + 4110806700\alpha^3\beta + 17547279750\alpha^2\beta^2 \\
&\quad + 27072945900\alpha\beta^3 + 13720932225\beta^4 + 2966303340\alpha^3 \\
&\quad + 38121583500\alpha^2\beta + 108613404900\alpha\beta^2 + 83489105700\beta^3 \\
&\quad + 15959803860\alpha^2 + 131075544600\alpha\beta + 182137655700\beta^2 \\
&\quad + 42567164640\alpha + 161880919200\beta + 45161035920, \\
Q_5^8(\alpha,\beta) &= 178378200\alpha^3 + 1967565600\alpha^2\beta + 5297292000\alpha\beta^2 + 3972969000\beta^3 \\
&\quad + 1418917500\alpha^2 + 11124313200\alpha\beta + 15305390100\beta^2 + 4589184600\alpha \\
&\quad + 17897279400\beta + 5708102400, \\
Q_6^8(\alpha,\beta) &= 77567490\alpha^2 + 510810300\alpha\beta + 652702050\beta^2 + 359459100\alpha \\
&\quad + 1305404100\beta + 522161640, \\
Q_7^8(\alpha,\beta) &= 18918900\alpha + 37837800 + 56756700\beta,\ Q_8^8(\alpha,\beta) = 2027025.
\end{aligned}$$

B.0.2 Jacobi Polynomials Case $m = 9$

For $m = 9$ we have

$$\mathscr{R}_9(X) = \sum_{j=1}^{9} (-1)^j \frac{Q_j^9(\alpha,\beta)}{2^9 \mathscr{A}_9(\alpha)} X^j \tag{B.18}$$

where $\mathscr{A}_9(\alpha) = (\alpha+1)(\alpha+2)\cdots(\alpha+9)$ and

$$\begin{aligned}
Q_1^9(\alpha,\beta) &= 819786496\alpha^8 + 55994442240\alpha^7\beta + 712037291520\alpha^6\beta^2 \\
&\quad + 3751484405760\alpha^5\beta^3 + 10310701040640\alpha^4\beta^4 + 16108264972800\alpha^3\beta^5 \\
&\quad + 14449801766400\alpha^2\beta^6 + 6947020080000\alpha\beta^7 + 1389404016000\beta^8 \\
&\quad + 36070605824\alpha^7 + 1639729282560\alpha^6\beta + 16515001152000\alpha^5\beta^2 \\
&\quad + 69437747957760\alpha^4\beta^3 + 148238381491200\alpha^3\beta^4 \\
&\quad + 169851332851200\alpha^2\beta^5 + 99759208348800\alpha\beta^6 + 23619868272000\beta^7 \\
&\quad + 677143645696\alpha^6 + 21281390584320\alpha^5\beta + 164535156042240\alpha^4\beta^2 \\
&\quad + 527333023526400\alpha^3\beta^3 + 816334405286400\alpha^2\beta^4 \\
&\quad + 607707673372800\alpha\beta^5 + 174787025212800\beta^6 + 7069838741504\alpha^5 \\
&\quad + 155516712552960\alpha^4\beta + 887584624243200\alpha^3\beta^2 \\
&\quad + 2028913737177600\alpha^2\beta^3 + 2019774412656000\alpha\beta^4
\end{aligned}$$

$$
\begin{aligned}
&+ 731104393219200\beta^5 + 44800512219904\alpha^4 + 681175344552960\alpha^3\beta \\
&+ 2700808505748480\alpha^2\beta^2 + 3913515084810240\alpha\beta^3 \\
&+ 1877029249455360\beta^4 + 175988485815296\alpha^3 + 1769425821911040\alpha^2\beta \\
&+ 4351258578700800\alpha\beta^2 + 3000081989399040\beta^3 + 417274605609984\alpha^2 \\
&+ 2502865890478080\alpha\beta + 2875094985692160\beta^2 + 544089018249216\alpha \\
&+ 1476524684759040\beta + 297484123668480, \\
Q_2^9(\alpha,\beta) &= 4903976400\alpha^7 + 227829619440\alpha^6\beta + 2351168099280\alpha^5\beta^2 \\
&+ 10140848237520\alpha^4\beta^3 + 22191465376080\alpha^3\beta^4 + 26028499376880\alpha^2\beta^5 \\
&+ 15625833022800\alpha\beta^6 + 3776201629200\beta^7 + 154876542960\alpha^6 \\
&+ 5389032561120\alpha^5\beta + 44110208142000\alpha^4\beta^2 + 147732829564320\alpha^3\beta^3 \\
&+ 237304141074000\alpha^2\beta^4 + 182432717026560\alpha\beta^5 + 53993232493200\beta^6 \\
&+ 2256078680880\alpha^5 + 56416336210800\alpha^4\beta + 347027520359520\alpha^3\beta^2 \\
&+ 838796363364960\alpha^2\beta^3 + 873794605123440\alpha\beta^4 + 328731997754160\beta^5 \\
&+ 18754423208400\alpha^4 + 326050490488320\alpha^3\beta + 1406127392429760\alpha^2\beta^2 \\
&+ 2170288182142080\alpha\beta^3 + 1095578478474480\beta^4 + 93726270522240\alpha^3 \\
&+ 1076572082226240\alpha^2\beta + 2892891334132800\alpha\beta^2 \\
&+ 2134357459153920\beta^3 + 277864422571200\alpha^2 + 1899666989625600\alpha\beta \\
&+ 2390366396658240\beta^2 + 448853359023360\alpha + 1385647230401280\beta \\
&+ 303030271272960, \\
Q_3^9(\alpha,\beta) &= 12048424620\alpha^6 + 392673710520\alpha^5\beta + 3183233032980\alpha^4\beta^2 \\
&+ 10719582793920\alpha^3\beta^3 + 17409921508980\alpha^2\beta^4 + 13562402653800\alpha\beta^5 \\
&+ 4070485118700\beta^6 + 278204214600\alpha^5 + 7227716113800\alpha^4\beta \\
&+ 45798027715440\alpha^3\beta^2 + 113743559769840\alpha^2\beta^3 \\
&+ 121498804987560\alpha\beta^4 + 46782391055400\beta^5 + 3054642410220\alpha^4 \\
&+ 57876636767040\alpha^3\beta + 263234408116680\alpha^2\beta^2 + 423515044592640\alpha\beta^3 \\
&+ 221402605083660\beta^4 + 19197754963920\alpha^3 + 244674585202320\alpha^2\beta \\
&+ 702403974352080\alpha\beta^2 + 545129908923600\beta^3 + 69982221978960\alpha^2 \\
&+ 533641686308640\alpha\beta + 722278038802320\beta^2 + 137138207843520\alpha \\
&+ 472461001346880\beta + 111390110012160,
\end{aligned}
$$

$$
\begin{aligned}
Q_4^9(\alpha,\beta) &= 16216118100\alpha^5 + 374045571900\alpha^4\beta + 2286267984000\alpha^3\beta^2 \\
&\quad + 5615507898000\alpha^2\beta^3 + 5994699410700\alpha\beta^4 + 2318567951700\beta^5 \\
&\quad + 272187131580\alpha^4 + 5108283540360\alpha^3\beta + 23409583076880\alpha^2\beta^2 \\
&\quad + 38163123999000\alpha\beta^3 + 20255112177300\beta^4 + 2165072167440\alpha^3 \\
&\quad + 29071698495840\alpha^2\beta + 86484629190960\alpha\beta^2 + 69155930844000\beta^3 \\
&\quad + 9582205127040\alpha^2 + 79140820318560\alpha\beta + 112813651510560\beta^2 \\
&\quad + 22449567722400\alpha + 84653740211040\beta + 21595548723840, \\
Q_5^9(\alpha,\beta) &= 13285256985\alpha^4 + 214061948100\alpha^3\beta + 926751775950\alpha^2\beta^2 \\
&\quad + 1473347175300\alpha\beta^3 + 773579631825\beta^4 + 157364707500\alpha^3 \\
&\quad + 2020011493500\alpha^2\beta + 5946456215700\alpha\beta^2 + 4761603346500\beta^3 \\
&\quad + 850798608660\alpha^2 + 7208084683800\alpha\beta + 10496203017300\beta^2 \\
&\quad + 2347504679520\alpha + 9404941946400\beta + 2635567094160, \\
Q_6^9(\alpha,\beta) &= 6873506640\alpha^3 + 74101547520\alpha^2\beta + 202621419000\alpha\beta^2 \\
&\quad + 156307951800\beta^3 + 54004810860\alpha^2 + 426083898240\alpha\beta \\
&\quad + 604969665300\beta^2 + 176230094040\alpha + 709753884840\beta \\
&\quad + 226715448960, \\
Q_7^9(\alpha,\beta) &= 2219186970\alpha^2 + 14472958500\alpha\beta + 18814846050\beta^2 \\
&\quad + 10227557340\alpha + 37629692100\beta + 15051876840, \\
Q_8^9(\alpha,\beta) &= 413513100\alpha + 827026200 + 1240539300\beta,\ Q_9^9(\alpha,\beta) = 34459425.
\end{aligned}
$$

B.0.3 Jacobi Poylnomials Case $m = 10$

For $m = 10$ we can write

$$\mathscr{R}_{10}(X) = \sum_{j=1}^{10} (-1)^j \frac{Q_j^{10}(\alpha,\beta)}{2^{10}\mathscr{A}_{10}(\alpha)} \tag{B.19}$$

where $\mathscr{A}_{10}(\alpha) = (\alpha+1)(\alpha+2)\cdots(\alpha+10)$ and

$$
\begin{aligned}
Q_1^{10}(\alpha,\beta) &= 56814228736\alpha^9 + 4778633088768\alpha^8\beta + 74633142796800\alpha^7\beta^2 \\
&\quad + 487475148142080\alpha^6\beta^3 + 1695092146675200\alpha^5\beta^4 \\
&\quad + 3469875535526400\alpha^4\beta^5 + 4330639993680000\alpha^3\beta^6 \\
&\quad + 3247037185392000\alpha^2\beta^7 + 1346332491504000\alpha\beta^8
\end{aligned}
$$

$$
\begin{aligned}
& + 237588086736000\beta^9 + 3067968351744\alpha^8 + 173855777819136\alpha^7\beta \\
& + 2194598552616960\alpha^6\beta^2 + 11765828786342400\alpha^5\beta^3 \\
& + 33104930511744000\alpha^4\beta^4 + 53016168302697600\alpha^3\beta^5 \\
& + 48739498936128000\alpha^2\beta^6 + 23996396760336000\alpha\beta^7 \\
& + 4910153792544000\beta^8 + 71926813579776\alpha^7 \\
& + 2870697948389376\alpha^6\beta + 28589508552998400\alpha^5\beta^2 \\
& + 121665277240012800\alpha^4\beta^3 + 264730230522576000\alpha^3\beta^4 \\
& + 309838665815078400\alpha^2\beta^5 + 185966103763440000\alpha\beta^6 \\
& + 44983344422016000\beta^7 + 959251437978624\alpha^6 \\
& + 27526468286446080\alpha^5\beta + 210622160970009600\alpha^4\beta^2 \\
& + 681566145927408000\alpha^3\beta^3 + 1072543629489696000\alpha^2\beta^4 \\
& + 813643323782889600\alpha\beta^5 + 238651576267104000\beta^6 \\
& + 8004499872386304\alpha^5 + 165240797441217792\alpha^4\beta \\
& + 936167403555283200\alpha^3\beta^2 + 2159201976508321920\alpha^2\beta^3 \\
& + 2181898655068204800\alpha\beta^4 + 803594562891120000\beta^5 \\
& + 43245059230066176\alpha^4 + 629316361946517504\alpha^3\beta \\
& + 2485978704659063040\alpha^2\beta^2 + 3635675466101280000\alpha\beta^3 \\
& + 1768886846423424000\beta^4 + 150885183364834304\alpha^3 \\
& + 1473118313268854784\alpha^2\beta + 3622232463358118400\alpha\beta^2 \\
& + 2522516187223872000\beta^3 + 326894073190557696\alpha^2 \\
& + 1924898840584796160\alpha\beta + 2218125776864256000\beta^2 \\
& + 397690510875402240\alpha + 1068445840289587200\beta \\
& + 206167473237196800, \\
Q_2^{10}(\alpha,\beta) = {} & 352085816208\alpha^8 + 20473784409120\alpha^7\beta + 264330028421760\alpha^6\beta^2 \\
& + 1450984640807520\alpha^5\beta^3 + 4178293093054080\alpha^4\beta^4 \\
& + 6841091256199200\alpha^3\beta^5 + 6422043696868800\alpha^2\beta^6 \\
& + 3224598310533600\alpha\beta^7 + 672129154897200\beta^8 + 13881915170256\alpha^7 \\
& + 614519453713200\alpha^6\beta + 6468167894464080\alpha^5\beta^2 \\
& + 28713972590175600\alpha^4\beta^3 + 64735622501069520\alpha^3\beta^4 \\
& + 78151489859242800\alpha^2\beta^5 + 48222367723530000\alpha\beta^6
\end{aligned}
$$

$$
\begin{aligned}
&+ 11960203142487600\beta^7 + 258002950329936\alpha^6 \\
&+ 8413184202299520\alpha^5\beta + 69251606294077200\alpha^4\beta^2 \\
&+ 236554243972300800\alpha^3\beta^3 + 388996445279922480\alpha^2\beta^4 \\
&+ 306362421075110400\alpha\beta^5 + 92855175436623600\beta^6 \\
&+ 2815916022136560\alpha^5 + 66343336954465200\alpha^4\beta \\
&+ 407974868796096000\alpha^3\beta^2 + 1000561231249070400\alpha^2\beta^3 \\
&+ 1062703960272274320\alpha\beta^4 + 408249777901165200\beta^5 \\
&+ 19262634251146272\alpha^4 + 319334557189663680\alpha^3\beta \\
&+ 1375240069869133440\alpha^2\beta^2 + 2148377186458601280\alpha\beta^3 \\
&+ 1103061084029704800\beta^4 + 83492320687414464\alpha^3 \\
&+ 925807740971668800\alpha^2\beta + 2487309137544611520\alpha\beta^2 \\
&+ 1855787986124654400\beta^3 + 222248152961890944\alpha^2 \\
&+ 1482551192102561280\alpha\beta + 1869156729701654400\beta^2 \\
&+ 330442369574607360\alpha + 1004118444529804800\beta \\
&+ 209203169566003200,
\end{aligned}
$$

$$
\begin{aligned}
Q_3^{10}(\alpha,\beta) &= 908998105260\alpha^7 + 37812695347980\alpha^6\beta + 393814618224300\alpha^5\beta^2 \\
&+ 1755605382230700\alpha^4\beta^3 + 3997350594537300\alpha^3\beta^4 \\
&+ 4885302895773300\alpha^2\beta^5 + 3054428986108500\alpha\beta^6 \\
&+ 767800786252500\beta^7 + 26794117064400\alpha^6 + 910281491728560\alpha^5\beta \\
&+ 7711733695236480\alpha^4\beta^2 + 27034158090139200\alpha^3\beta^3 \\
&+ 45534353935870800\alpha^2\beta^4 + 36669049893075600\alpha\beta^5 \\
&+ 11346842469362400\beta^6 + 386667988105980\alpha^5 \\
&+ 9938616241724100\alpha^4\beta + 64385484661035720\alpha^3\beta^2 \\
&+ 164397294007947000\alpha^2\beta^3 + 180624840556059900\alpha\beta^4 \\
&+ 71474109679872900\beta^5 + 3325091707303800\alpha^4 \\
&+ 61137377055568800\alpha^3\beta + 280913024219471280\alpha^2\beta^2 \\
&+ 461032049043165600\alpha\beta^3 + 246532131116271000\beta^4 \\
&+ 17684952939499440\alpha^3 + 218421777503729520\alpha^2\beta \\
&+ 630247699319631120\alpha\beta^2 + 496798103579778000\beta^3 \\
&+ 56903399087892000\alpha^2 + 422759405853771840\alpha\beta
\end{aligned}
$$

$$
\begin{aligned}
&+ 5744189629033128000\beta^2 + 1012829911858195200\alpha\\
&+ 3423488205826080000\beta + 763281029734272000,\\
Q_4^{10}(\alpha,\beta) &= 1308596437680\alpha^6 + 39545477407800\alpha^5\beta + 323024120218800\alpha^4\beta^2\\
&+ 1118979129668400\alpha^3\beta^3 + 1882037058501600\alpha^2\beta^4\\
&+ 1521302289507000\alpha\beta^5 + 473814336996000\beta^6 + 28925722973700\alpha^5\\
&+ 738399720813300\alpha^4\beta + 4817606528334600\alpha^3\beta^2\\
&+ 12452962719997800\alpha^2\beta^3 + 13878640518642900\alpha\beta^4\\
&+ 5574088358338500\beta^5 + 315743221022460\alpha^4\\
&+ 6125130648327600\alpha^3\beta + 29142502689900600\alpha^2\beta^2\\
&+ 49229749591822800\alpha\beta^3 + 27005178035335500\beta^4\\
&+ 2042061833805840\alpha^3 + 27321344007375600\alpha^2\beta\\
&+ 82945217108754000\alpha\beta^2 + 68020851168270000\beta^3\\
&+ 7854008615209680\alpha^2 + 63823453064935200\alpha\beta\\
&+ 92036627909226000\beta^2 + 16528261328288640\alpha\\
&+ 61274070421200000\beta + 14613015545049600,\\
Q_5^{10}(\alpha,\beta) &= 1174524439725\alpha^5 + 25758569511675\alpha^4\beta + 158743429094250\alpha^3\beta^2\\
&+ 399967444626750\alpha^2\beta^3 + 440734150481625\alpha\beta^4\\
&+ 176334908124375\beta^5 + 19180744578450\alpha^4 + 356385621992400\alpha^3\beta\\
&+ 1677329465912100\alpha^2\beta^2 + 2835343543032000\alpha\beta^3\\
&+ 1564455482840250\beta^4 + 152327044478700\alpha^3\\
&+ 2092670385705900\alpha^2\beta + 6485108074344900\alpha\beta^2\\
&+ 5421667429678500\beta^3 + 692266081689000\alpha^2\\
&+ 5976908239302000\alpha\beta + 8964566208397800\beta^2\\
&+ 1702353052436400\alpha + 6800833274835600\beta + 1746421001488800,\\
Q_6^{10}(\alpha,\beta) &= 689604310395\alpha^4 + 10761127076700\alpha^3\beta + 47050140787650\alpha^2\beta^2\\
&+ 76611158523900\alpha\beta^3 + 41426672762475\beta^4 + 8032041697680\alpha^3\\
&+ 102844843101000\alpha^2\beta + 310624424510400\alpha\beta^2 + 257277099479400\beta^3\\
&+ 43538785109820\alpha^2 + 377810103270600\alpha\beta + 571649193215100\beta^2\\
&+ 123320229112080\alpha + 515371300844400\beta + 144909876711600,
\end{aligned}
$$

$$
\begin{aligned}
Q_7^{10}(\alpha,\beta) &= 267565948650\alpha^3 + 2832357978450\alpha^2\beta + 7837107027750\alpha\beta^2 \\
&\quad + 6187189758750\beta^3 + 2082038458500\alpha^2 + 16499172690000\alpha\beta \\
&\quad + 24033794885100\beta^2 + 6838863431400\alpha + 28268582542200\beta \\
&\quad + 9040499067600, \\
Q_8^{10}(\alpha,\beta) &= 66782365650\alpha^2 + 432121189500\alpha\beta + 569614295250\beta^2 \\
&\quad + 306413207100\alpha + 1139228590500\beta + 455691436200, \\
Q_9^{10}(\alpha,\beta) &= 9820936125\alpha + 19641872250 + 29462808375\beta,\ Q_{10}^{10}(\alpha,\beta) = 654729075.
\end{aligned}
$$

Appendix C

Hypergeometric Series for $6 \leq m \leq 10$

We can obtain the hypergeometric series using (2.82) and (2.9) similarly to (2.21) for $6 \leq m \leq 10$. Indeed, choosing values of m, i and j and substituting them into (2.82) and (2.9) allows us to establish the coefficients b_i^m and $d_{j,i}$. The coefficients $\mathsf{d}_{j,i}$ can also be calculated using the product (2.10). The coefficients $\mathsf{c}_j^m(a,b,c)$ are obtained from b_i^m and $\mathsf{d}_{j,i}$ using the formula

$$\mathsf{c}_j^m(a,b,c) = (-1)^j \sum_{i=j}^{m} (-2)^{-i} \mathsf{b}_i^m \mathsf{d}_{j,i} \prod_{p=0}^{i-1} (c+p)^{-1}. \tag{C.1}$$

Once the coefficients $\mathsf{c}_j^m(a,b,c)$ have been established, substitution into the summation

$$\mathscr{R}_m(X;a,b,c) = \sum_{j=1}^{m} \mathsf{c}_j^m(a,b,c) X^j \tag{C.2}$$

gives us the hypergeometric series for suitable choice of m. The polynomials that emerge are of the form

$$\mathscr{R}_m(X;a,b,c) = \sum_{l=1}^{m} \mathsf{c}_l^m X^l = \sum_{l=1}^{m} \frac{\mathsf{q}_l^m(a,b,c)}{2^m \mathscr{A}_m(c)} X^l \tag{C.3}$$

where $\mathscr{A} = c(c+1)(c+2)\cdots(c+m-1)$ and $\mathsf{q}_l^m(a,b,c)$ is yet to be established.

For $m = 6$ we can write

$$\mathscr{R}_6(X;a,b,c) = \sum_{l=1}^{6} \mathsf{c}_l^6 X^l = \sum_{l=1}^{6} \frac{\mathsf{q}_l^6(a,b,c)}{2^6 \mathscr{A}_6(c)} X^l \tag{C.4}$$

where $\mathscr{A}_6(c) = c(c+1)(c+2)\cdots(c+5)$ and

$$\begin{aligned}
\mathsf{c}_1^6(a,b,c) &= -1247400a^5 - 6237000a^4b + 2494800a^4c - 12474000a^3b^2 + 9979200a^3bc \\
&\quad - 1580040a^3c^2 - 12474000a^2b^3 + 14968800a^2b^2c - 4740120a^2bc^2 + 337920a^2c^3 \\
&\quad - 6237000ab^4 + 9979200ab^3c - 4740120ab^2c^2 + 675840abc^3 - 16368ac^4 \\
&\quad - 1247400b^5 + 2494800b^4c - 1580040b^3c^2 + 337920b^2c^3 - 16368bc^4 + 32c^5 \\
&\quad - 6237000a^4 - 24948000a^3b + 10727640a^3c - 37422000a^2b^2 + 32182920a^2bc \\
&\quad - 5425200a^2c^2 - 24948000ab^3 + 32182920ab^2c - 10850400abc^2 + 784608ac^3 \\
&\quad - 6237000b^4 + 10727640b^3c - 5425200b^2c^2 + 784608bc^3 - 15888c^4 - 12889800a^3 \\
&\quad - 38669400a^2b + 17878080a^2c - 38669400ab^2 + 35756160abc - 6377448ac^2 \\
&\quad - 12889800b^3 + 17878080b^2c - 6377448bc^2 + 449408c^3 - 13404600a^2 \\
&\quad - 26809200ab + 13442088ac - 13404600b^2 + 13442088bc - 2525088c^2 \\
&\quad - 6834960a - 6834960b + 3805616c - 1326720, \\
\mathsf{c}_2^6(a,b,c) &= 2848230a^4 + 11392920a^3b - 5197500a^3c + 17089380a^2b^2 - 15592500a^2bc \\
&\quad + 2896740a^2c^2 + 11392920ab^3 - 15592500ab^2c + 5793480abc^2 - 506880ac^3 \\
&\quad + 2848230b^4 - 5197500b^3c + 2896740b^2c^2 - 506880bc^3 + 16368c^4 + 10498950a^3 \\
&\quad + 31496850a^2b - 15925140a^2c + 31496850ab^2 - 31850280abc + 6557760ac^2 \\
&\quad + 10498950b^3 - 15925140b^2c + 6557760bc^2 - 615648c^3 + 15211350a^2 \\
&\quad + 30422700ab - 17128980ac + 15211350b^2 - 17128980bc + 3928188c^2 \\
&\quad + 9815850a + 9815850b - 6290988c + 2239380, \\
\mathsf{c}_3^6(a,b,c) &= -2338875a^3 - 7016625a^2b + 3638250a^2c - 7016625ab^2 + 7276500abc \\
&\quad - 1580040ac^2 - 2338875b^3 + 3638250b^2c - 1580040bc^2 + 168960c^3 - 5498955a^2 \\
&\quad - 10997910ab + 6569640ac - 5498955b^2 + 6569640bc - 1659240c^2 - 4521825a \\
&\quad - 4521825b + 3138630c - 1175625, \\
\mathsf{c}_4^6(a,b,c) &= 883575a^2 + 1767150ab - 1039500ac + 883575b^2 - 1039500bc + 263340c^2 \\
&\quad + 1039500a + 1039500b - 748440c + 308385, \\
\mathsf{c}_5^6(a,b,c) &= 103950c - 51975 - 155925a - 155925b, \;\; \mathsf{c}_6^6(a,b,c) = 10395.
\end{aligned}$$

C.0.1 Hypergeometric Series Case $m = 7$

For $m = 7$ we can write

$$\mathscr{R}_7(X;a,b,c) = \sum_{l=1}^{7} \mathsf{c}_l^7 X^l = \sum_{l=1}^{7} \frac{\mathsf{q}_l^7(a,b,c)}{2^7 \mathscr{A}_7(c)} X^l \tag{C.5}$$

where $\mathscr{A}_7(c) = c(c+1)(c+2)\cdots(c+6)$ and

$$
\begin{aligned}
\mathsf{c}_1^7(a,b,c) = {} & -97297200a^6 - 583783200a^5b + 227026800a^5c - 1459458000a^4b^2 \\
& + 1135134000a^4bc - 181621440a^4c^2 - 1945944000a^3b^3 \\
& + 2270268000a^3b^2c - 726485760a^3bc^2 + 57657600a^3c^3 \\
& - 1459458000a^2b^4 + 2270268000a^2b^3c - 1089728640a^2b^2c^2 \\
& + 172972800a^2bc^3 - 6246240a^2c^4 - 583783200ab^5 + 1135134000ab^4c \\
& - 726485760ab^3c^2 + 172972800ab^2c^3 - 12492480abc^4 + 131040ac^5 \\
& - 97297200b^6 + 227026800b^5c - 181621440b^4c^2 + 57657600b^3c^3 \\
& - 6246240b^2c^4 + 131040bc^5 - 64c^6 - 681080400a^5 - 3405402000a^4b \\
& + 1407566160a^4c - 6810804000a^3b^2 + 5630264640a^3bc \\
& - 951350400a^3c^2 - 6810804000a^2b^3 + 8445396960a^2b^2c \\
& - 2854051200a^2bc^2 + 233513280a^2c^3 - 3405402000ab^4 \\
& + 5630264640ab^3c - 2854051200ab^2c^2 + 467026560abc^3 - 16117920ac^4 \\
& - 681080400b^5 + 1407566160b^4c - 951350400b^3c^2 + 233513280b^2c^3 \\
& - 16117920bc^4 + 129696c^5 - 2043241200a^4 - 8172964800a^3b \\
& + 3585582000a^3c - 12259447200a^2b^2 + 10756746000a^2bc \\
& - 1910868960a^2c^2 - 8172964800ab^3 + 10756746000ab^2c \\
& - 3821737920abc^2 + 317247840ac^3 - 2043241200b^4 + 3585582000b^3c \\
& - 1910868960b^2c^2 + 317247840bc^3 - 9882880c^4 - 3297294000a^3 \\
& - 9891882000a^2b + 4620535920a^2c - 9891882000ab^2 + 9241071840abc \\
& - 1721472480ac^2 - 3297294000b^3 + 4620535920b^2c - 1721472480bc^2 \\
& + 141345120c^3 - 2962159200a^2 - 5924318400ab + 2975001120ac \\
& - 2962159200b^2 + 2975001120bc - 580436416c^2 - 1375920000a \\
& - 1375920000b + 759394944c - 251642880, \\
\mathsf{c}_2^7(a,b,c) = {} & 238378140a^5 + 1191890700a^4b - 518377860a^4c + 2383781400a^3b^2 \\
& - 2073511440a^3bc + 378378000a^3c^2 + 2383781400a^2b^3 \\
& - 3110267160a^2b^2c + 1135134000a^2bc^2 - 105705600a^2c^3 \\
& + 1191890700ab^4 - 2073511440ab^3c + 1135134000ab^2c^2 \\
& - 211411200abc^3 + 9369360ac^4 + 238378140b^5 - 518377860b^4c
\end{aligned}
$$

$$
\begin{aligned}
& + 378378000b^3c^2 - 105705600b^2c^3 + 9369360bc^4 - 131040c^5 \\
& + 1311890580a^4 + 5247562320a^3b - 2478375900a^3c + 7871343480a^2b^2 \\
& - 7435127700a^2bc + 1471710240a^2c^2 + 5247562320ab^3 \\
& - 7435127700ab^2c + 2943420480abc^2 - 292612320ac^3 + 1311890580b^4 \\
& - 2478375900b^3c + 1471710240b^2c^2 - 292612320bc^3 + 12994800c^4 \\
& + 3008105100a^3 + 9024315300a^2b - 4632487860a^2c + 9024315300ab^2 \\
& - 9264975720abc + 1990628640ac^2 + 3008105100b^3 - 4632487860b^2c \\
& + 1990628640bc^2 - 210100800c^3 + 3490987500a^2 + 6981975000ab \\
& - 3931707780ac + 3490987500b^2 - 3931707780bc + 924596400c^2 \\
& + 1985943960a + 1985943960b - 1254762600c + 421686720, \\
\mathsf{c}_3^7(a,b,c) &= -219459240a^4 - 877836960a^3b + 425675250a^3c - 1316755440a^2b^2 \\
& + 1277025750a^2bc - 264864600a^2c^2 - 877836960ab^3 \\
& + 1277025750ab^2c - 529729200abc^2 + 57657600ac^3 - 219459240b^4 \\
& + 425675250b^3c - 264864600b^2c^2 + 57657600bc^3 - 3123120c^4 \\
& - 862701840a^3 - 2588105520a^2b + 1398106710a^2c - 2588105520ab^2 \\
& + 2796213420abc - 648648000ac^2 - 862701840b^3 + 1398106710b^2c \\
& - 648648000bc^2 + 78318240c^3 - 1333512180a^2 - 2667024360ab \\
& + 1608556950ac - 1333512180b^2 + 1608556950bc - 419579160c^2 \\
& - 913512600a - 913512600b + 625554930c - 217477260, \\
\mathsf{c}_4^7(a,b,c) &= 99324225a^3 + 297972675a^2b - 160810650a^2c + 297972675ab^2 \\
& - 321621300abc + 75675600ac^2 + 99324225b^3 - 160810650b^2c \\
& + 75675600bc^2 - 9609600c^3 + 243107865a^2 + 486215730ab \\
& - 302702400ac + 243107865b^2 - 302702400bc + 82882800c^2 \\
& + 207161955a + 207161955b - 149219070c + 55090035, \\
\mathsf{c}_5^7(a,b,c) &= -23648625a^2 - 47297250ab + 28378350ac - 23648625b^2 \\
& + 28378350bc - 7567560c^2 - 28378350a - 28378350b + 20810790c \\
& - 8513505, \\
\mathsf{c}_6^7(a,b,c) &= -1891890c + 945945 + 2837835a + 2837835b, \\
\mathsf{c}_7^7(a,b,c) &= -135135.
\end{aligned}
$$

C.0.2 Hypergeometric Series Case $m = 8$

For $m = 8$ we can write

$$\mathscr{R}_8(X; a, b, c) = \sum_{l=1}^{8} \mathsf{c}_l^8 X^l = \sum_{l=1}^{8} \frac{\mathsf{q}_l^8(a, b, c)}{2^8 \mathscr{A}_8(c)} X^l \tag{C.6}$$

where $\mathscr{A}_8(c) = c(c + 1)(c + 2) \cdots (c + 7)$ and

$$\begin{aligned}
\mathsf{c}_1^8(a, b, c) = {} & -10216206000a^7 - 71513442000a^6b + 27243216000a^6 \\
& - 214540326000a^5b^2 + 163459296000a^5bc - 26335108800a^5c^2 \\
& - 357567210000a^4b^3 + 408648240000a^4b^2c - 131675544000a^4bc^2 \\
& + 11156745600a^4c^3 - 357567210000a^3b^4 + 544864320000a^3b^3c \\
& - 263351088000a^3b^2c^2 + 44626982400a^3bc^3 - 1976214240a^3c^4 \\
& - 214540326000a^2b^5 + 408648240000a^2b^4c - 263351088000a^2b^3c^2 \\
& + 66940473600a^2b^2c^3 - 5928642720a^2bc^4 + 113742720a^2c^5 \\
& - 71513442000ab^6 + 163459296000ab^5c - 131675544000ab^4c^2 \\
& + 44626982400ab^3c^3 - 5928642720ab^2c^4 + 227485440abc^5 \\
& - 1048512ac^6 - 10216206000b^7 + 27243216000b^6c - 26335108800b^5c^2 \\
& + 11156745600b^4c^3 - 1976214240b^3c^4 + 113742720b^2c^5 - 1048512bc^6 \\
& + 128c^7 - 95351256000a^6 - 572107536000a^5b + 229751121600a^5c \\
& - 1430268840000a^4b^2 + 1148755608000a^4bc - 194205211200a^4c^2 \\
& - 1907025120000a^3b^3 + 2297511216000a^3b^2c - 776820844800a^3bc^2 \\
& + 68090742720a^3c^3 - 1430268840000a^2b^4 + 2297511216000a^2b^3c \\
& - 1165231267200a^2b^2c^2 + 204272228160a^2bc^3 - 9013717440a^2c^4 \\
& - 572107536000ab^5 + 1148755608000ab^4c - 776820844800ab^3c^2 \\
& + 204272228160ab^2c^3 - 18027434880abc^4 + 312918336ac^5 \\
& - 95351256000b^6 + 229751121600b^5c - 194205211200b^4c^2 \\
& + 68090742720b^3c^3 - 9013717440b^2c^4 + 312918336bc^5 - 1044928c^6 \\
& - 390940149600a^5 - 1954700748000a^4b + 825988363200a^4c \\
& - 3909401496000a^3b^2 + 3303953452800a^3bc - 584012388960a^3c^2 \\
& - 3909401496000a^2b^3 + 4955930179200a^2b^2c - 1752037166880a^2bc^2 \\
& + 157492782720a^2c^3 - 1954700748000ab^4 + 3303953452800ab^3c
\end{aligned}$$

$$
\begin{aligned}
&- 1752037166880ab^2c^2 + 314985565440abc^3 - 13516963680ac^4 \\
&- 390940149600b^5 + 825988363200b^4c - 584012388960b^3c^2 \\
&+ 157492782720b^2c^3 - 13516963680bc^4 + 199216832c^5 \\
&- 899026128000a^4 - 3596104512000a^3b + 1600361642880a^3c \\
&- 5394156768000a^2b^2 + 4801084928640a^2bc - 885456149760a^2c^2 \\
&- 3596104512000ab^3 + 4801084928640ab^2c - 1770912299520abc^2 \\
&+ 161448560640ac^3 - 899026128000b^4 + 1600361642880b^3c \\
&- 885456149760b^2c^2 + 161448560640bc^3 - 6479209600c^4 \\
&- 1234344711600a^3 - 3703034134800a^2b + 1743101156160a^2c \\
&- 3703034134800ab^2 + 3486202312320abc - 670323127008ac^2 \\
&- 1234344711600b^3 + 1743101156160b^2c - 670323127008bc^2 \\
&+ 60890641472c^3 - 996523819200a^2 - 1993047638400ab \\
&+ 1002217800384ac - 996523819200b^2 + 1002217800384bc \\
&- 201007582912c^2 - 430361225280a - 430361225280b \\
&+ 235999502208c - 74960444160, \\
\mathsf{c}_2^8(a,b,c) &= 26489162700a^6 + 158934976200a^5b - 66745879200a^5c \\
&+ 397337440500a^4b^2 - 333729396000a^4bc + 60131831760a^4c^2 \\
&+ 529783254000a^3b^3 - 667458792000a^3b^2c + 240527327040a^3bc^2 \\
&- 23243220000a^3c^3 + 397337440500a^2b^4 - 667458792000a^2b^3c \\
&+ 360790990560a^2b^2c^2 - 69729660000a^2bc^3 + 3623059440a^2c^4 \\
&+ 158934976200ab^5 - 333729396000ab^4c + 240527327040ab^3c^2 \\
&- 69729660000ab^2c^3 + 7246118880abc^4 - 170614080ac^5 \\
&+ 26489162700b^6 - 66745879200b^5c + 60131831760b^4c^2 \\
&- 23243220000b^3c^3 + 3623059440b^2c^4 - 170614080bc^5 + 1048512c^6 \\
&+ 202961959200a^5 + 1014809796000a^4b - 456490354320a^4c \\
&+ 2029619592000a^3b^2 - 1825961417280a^3bc + 351923972400a^3c^2 \\
&+ 2029619592000a^2b^3 - 2738942125920a^2b^2c + 1055771917200a^2bc^2 \\
&- 108097909920a^2c^3 + 1014809796000ab^4 - 1825961417280ab^3c
\end{aligned}
$$

$$
\begin{aligned}
&+ 1055771917200ab^2c^2 - 216195819840abc^3 + 11544361920ac^4 \\
&+ 202961959200b^5 - 456490354320b^4c + 351923972400b^3c^2 \\
&- 108097909920b^2c^3 + 11544361920bc^4 - 256046976c^5 \\
&+ 671295544920a^4 + 2685182179680a^3b - 1293090598800a^3c \\
&+ 4027773269520a^2b^2 - 3879271796400a^2bc + 799132894560a^2c^2 \\
&+ 2685182179680ab^3 - 3879271796400ab^2c + 1598265789120abc^2 \\
&- 172797075360ac^3 + 671295544920b^4 - 1293090598800b^3c \\
&+ 799132894560b^2c^2 - 172797075360bc^3 + 9339474000c^4 \\
&+ 1202333932800a^3 + 3607001798400a^2b - 1867327141680a^2c \\
&+ 3607001798400ab^2 - 3734654283360abc + 825090673680ac^2 \\
&+ 1202333932800b^3 - 1867327141680b^2c + 82509067368bc^2 \\
&- 94050626400c^3 + 1204736633100a^2 + 2409473266200ab \\
&- 1355273534160ac + 1204736633100b^2 - 1355273534160bc \\
&+ 324052613328c^2 + 623910924000a + 623910924000b \\
&- 389735792064c + 125035338960, \\
\mathsf{c}_3^8(a,b,c) &= -26618892300a^5 - 133094461500a^4b + 61448587200a^4c \\
&- 266188923000a^3b^2 + 245794348800a^3bc - 49378329000a^3c^2 \\
&- 266188923000a^2b^3 + 368691523200a^2b^2c - 148134987000a^2bc^2 \\
&+ 16270254000a^2c^3 - 133094461500ab^4 + 245794348800ab^3c \\
&- 148134987000ab^2c^2 + 32540508000abc^3 - 1976214240ac^4 \\
&- 26618892300b^5 + 61448587200b^4c - 49378329000b^3c^2 \\
&+ 16270254000b^2c^3 - 1976214240bc^4 + 56871360c^5 - 156253897800a^4 \\
&- 625015591200a^3b + 314772658200a^3c - 937523386800a^2b^2 \\
&+ 944317974600a^2bc - 207283035960a^2c^2 - 625015591200ab^3 \\
&+ 944317974600ab^2c - 414566071920abc^2 + 49496166720ac^3 \\
&- 156253897800b^4 + 314772658200b^3c - 207283035960b^2c^2 \\
&+ 49496166720bc^3 - 3189382560c^4 - 382880698200a^3 \\
&- 1148642094600a^2b + 631036125720a^2c - 1148642094600ab^2
\end{aligned}
$$

$$+ 1262072251440abc - 303131588760ac^2 - 382880698200b^3$$
$$+ 631036125720b^2c - 303131588760bc^2 + 39395800080c^3$$
$$- 474652498320a^2 - 949304996640ab + 573222129480ac$$
$$- 474652498320b^2 + 573222129480bc - 151997194440c^2$$
$$- 287018631900a - 287018631900b + 194085669960c - 63770452200,$$

$$\begin{aligned}
\mathsf{c}_4^8(a,b,c) &= 13720932225a^4 + 54883728900a^3b - 27810783000a^3c \\
&+ 82325593350a^2b^2 - 83432349000a^2bc + 18654035400a^2c^2 \\
&+ 54883728900ab^3 - 83432349000ab^2c + 37308070800abc^2 \\
&- 4648644000ac^3 + 13720932225b^4 - 27810783000b^3c \\
&+ 18654035400b^2c^2 - 4648644000bc^3 + 329369040c^4 + 56416159800a^3 \\
&+ 169248479400a^2b - 95729634000a^2c + 169248479400ab^2 \\
&- 191459268000abc + 47999952000ac^2 + 56416159800b^3 \\
&- 95729634000b^2c + 47999952000bc^2 - 6699813120c^3 \\
&+ 91071530550a^2 + 182143061100ab - 114978263400ac \\
&+ 91071530550b^2 - 114978263400bc + 32437084680c^2 \\
&+ 64816151400a + 64816151400b - 46249323120c + 15832281465,
\end{aligned}$$

$$\begin{aligned}
\mathsf{c}_5^8(a,b,c) &= -3972969000a^3 - 11918907000a^2b + 6621615000a^2c \\
&- 11918907000ab^2 + 13243230000abc - 3291888600ac^2 \\
&- 3972969000b^3 + 6621615000b^2c - 3291888600bc^2 + 464864400c^3 \\
&- 10008098100a^2 - 20016196200ab + 12827014200ac - 10008098100b^2 \\
&+ 12827014200bc - 3702699000c^2 - 8740531800a - 8740531800b \\
&+ 6454047600c - 2359457100,
\end{aligned}$$

$$\begin{aligned}
\mathsf{c}_6^8(a,b,c) &= 652702050a^2 + 1305404100ab - 794593800ac + 652702050b^2 \\
&- 794593800bc + 219459240c^2 + 794593800a + 794593800b \\
&- 590269680c + 240270030,
\end{aligned}$$

$$\mathsf{c}_7^8(a,b,c) = 37837800c - 18918900 - 56756700a - 56756700b,$$

$$\mathsf{c}_8^8(a,b,c) = 2027025.$$

C.0.3 Hypergeometric Series Case $m = 9$

For $m = 9$ we can write

$$\mathscr{R}_9(X; a, b, c) = \sum_{l=1}^{9} \mathsf{c}_l^9 X^l = \sum_{l=1}^{9} \frac{\mathsf{q}_l^9(a, b, c)}{2^9 \mathscr{A}_9(c)} X^l \tag{C.7}$$

where $\mathscr{A}_9(c) = c(c+1)(c+2)\cdots(c+8)$ and

$$\begin{aligned}
\mathsf{c}_1^9(a, b, c) = & -1389404016000a^8 - 11115232128000a^7b + 4168212048000a^7c \\
& - 38903312448000a^6b^2 + 29177484336000a^6bc - 4723973654400a^6c^2 \\
& - 77806624896000a^5b^3 + 87532453008000a^5b^2c \\
& - 28343841926400a^5bc^2 + 2509748841600a^5c^3 - 97258281120000a^4b^4 \\
& + 145887421680000a^4b^3c - 70859604816000a^4b^2c^2 \\
& + 12548744208000a^4bc^3 - 628980992640a^4c^4 - 77806624896000a^3b^5 \\
& + 145887421680000a^3b^4c - 94479473088000a^3b^3c^2 \\
& + 25097488416000a^3b^2c^3 - 2515923970560a^3bc^4 + 65627452800a^3c^5 \\
& - 38903312448000a^2b^6 + 87532453008000a^2b^5c \\
& - 70859604816000a^2b^4c^2 + 25097488416000a^2b^3c^3 \\
& - 3773885955840a^2b^2c^4 + 196882358400a^2bc^5 - 2057854080a^2c^6 \\
& - 11115232128000ab^7 + 29177484336000ab^6c - 28343841926400ab^5c^2 \\
& + 12548744208000ab^4c^3 - 2515923970560ab^3c^4 + 196882358400ab^2c^5 \\
& - 4115708160abc^6 + 8388480ac^7 - 1389404016000b^8 \\
& + 4168212048000b^7c - 4723973654400b^6c^2 + 2509748841600b^5c^3 \\
& - 628980992640b^4c^4 + 65627452800b^3c^5 - 2057854080b^2c^6 \\
& + 8388480bc^7 - 256c^8 - 16672848192000a^7 - 116709937344000a^6b \\
& + 45850332528000a^6c - 350129812032000a^5b^2 \\
& + 275101995168000a^5bc - 46498721068800a^5c^2 \\
& - 583549686720000a^4b^3 + 687754987920000a^4b^2c \\
& - 232493605344000a^4bc^2 + 21292726815360a^4c^3 \\
& - 583549686720000a^3b^4 + 917006650560000a^3b^3c \\
& - 464987210688000a^3b^2c^2 + 85170907261440a^3bc^3 \\
& - 4320986342400a^3c^4 - 350129812032000a^2b^5
\end{aligned}$$

$$+ 687754987920000a^2b^4c - 464987210688000a^2b^3c^2$$
$$+ 127756360892160a^2b^2c^3 - 12962959027200a^2bc^4 + 325855532160a^2c^5$$
$$- 116709937344000ab^6 + 275101995168000ab^5c$$
$$- 232493605344000ab^4c^2 + 85170907261440ab^3c^3$$
$$- 12962959027200ab^2c^4 + 651711064320abc^5 - 5879965440ac^6$$
$$- 166728481920000b^7 + 458503325280000b^6c - 464987210688000b^5c^2$$
$$+ 212927268153600b^4c^3 - 43209863424000b^3c^4 + 325855532160b^2c^5$$
$$- 5879965440bc^6 + 8379264c^7 - 89477618630400a^6$$
$$- 536865711782400a^5b + 220535909193600a^5c$$
$$- 1342164279456000a^4b^2 + 1102679545968000a^4bc$$
$$- 193998733568640a^4c^2 - 1789552372608000a^3b^3$$
$$+ 2205359091936000a^3b^2c - 775994934274560a^3bc^2$$
$$+ 73091339193600a^3c^3 - 1342164279456000a^2b^4$$
$$+ 2205359091936000a^2b^3c - 1163992401411840a^2b^2c^2$$
$$+ 219274017580800a^2bc^3 - 11117138304000a^2c^4$$
$$- 536865711782400ab^5 + 1102679545968000ab^4c$$
$$- 775994934274560ab^3c^2 + 219274017580800ab^2c^3$$
$$- 22234276608000abc^4 + 522460940160ac^5 - 89477618630400b^6$$
$$+ 220535909193600b^5c - 193998733568640b^4c^2 + 73091339193600b^3c^3$$
$$- 11117138304000b^2c^4 + 522460940160bc^5 - 3822251136c^6$$
$$- 277139787724800a^5 - 1385698938624000a^4b + 595120996229760a^4c$$
$$- 2771397877248000a^3b^2 + 2380483984919040a^3bc$$
$$- 435112417958400a^3c^2 - 2771397877248000a^2b^3$$
$$+ 3570725977378560a^2b^2c - 1305337253875200a^2bc^2$$
$$+ 125655887980800a^2c^3 - 1385698938624000ab^4$$
$$+ 2380483984919040ab^3c - 1305337253875200ab^2c^2$$
$$+ 251311775961600abc^3 - 12505461772800ac^4 - 277139787724800b^5$$

$$
\begin{aligned}
&+ 595120996229760b^4c - 435112417958400b^3c^2 + 125655887980800b^2c^3\\
&- 12505461772800bc^4 + 262231699584c^5 - 535661561635200a^4\\
&- 2142646246540800a^3b + 963953135308800a^3c\\
&- 3213969369811200a^2b^2 + 2891859405926400a^2bc\\
&- 548686153240320a^2c^2 - 2142646246540800ab^3\\
&+ 2891859405926400ab^2c - 1097372306480640abc^2\\
&+ 107071527083520ac^3 - 535661561635200b^4 + 963953135308800b^3c\\
&- 548686153240320b^2c^2 + 107071527083520bc^3 - 5080334565504c^4\\
&- 653833881792000a^3 - 1961501645376000a^2b + 928944047600640a^2c\\
&- 1961501645376000ab^2 + 1857888095201280abc\\
&- 365992663503360ac^2 - 653833881792000b^3 + 928944047600640b^2c\\
&- 365992663503360bc^2 + 35723983097856c^3 - 485792480678400a^2\\
&- 971584961356800ab + 489150992931840ac - 485792480678400b^2\\
&+ 489150992931840bc - 100194972306944c^2 - 197728328448000a\\
&- 197728328448000b + 107892845070336c - 33053791518720,
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{c}_2^9(a,b,c) &= 3776201629200a^7 + 26433411404400a^6b - 10807578381600a^6c\\
&+ 79300234213200a^5b^2 - 64845470289600a^5bc + 11573735453280a^5c^2\\
&+ 132167057022000a^4b^3 - 162113675724000a^4b^2c\\
&+ 57868677266400a^4bc^2 - 5730593188320a^4c^3 + 132167057022000a^3b^4\\
&- 216151567632000a^3b^3c + 115737354532800a^3b^2c^2\\
&- 22922372753280a^3bc^3 + 1310377068000a^3c^4 + 79300234213200a^2b^5\\
&- 162113675724000a^2b^4c + 115737354532800a^2b^3c^2\\
&- 34383559129920a^2b^2c^3 + 3931131204000a^2bc^4 - 120316996800a^2c^5\\
&+ 26433411404400ab^6 - 64845470289600ab^5c + 57868677266400ab^4c^2\\
&- 22922372753280ab^3c^3 + 3931131204000ab^2c^4 - 240633993600abc^5\\
&+ 3086781120ac^6 + 3776201629200b^7 - 10807578381600b^6c\\
&+ 11573735453280b^5c^2 - 5730593188320b^4c^3 + 1310377068000b^3c^4
\end{aligned}
$$

$$- 120316996800b^2c^5 + 3086781120bc^6 - 8388480c^7$$
$$+ 38367399470400a^6 + 230204396822400a^5b - 99828678549600a^5c$$
$$+ 575510992056000a^4b^2 - 499143392748000a^4bc$$
$$+ 94362145637760a^4c^2 + 767347989408000a^3b^3$$
$$- 998286785496000a^3b^2c + 377448582551040a^3bc^2$$
$$- 39340349848800a^3c^3 + 575510992056000a^2b^4$$
$$- 998286785496000a^2b^3c + 566172873826560a^2b^2c^2$$
$$- 118021049546400a^2bc^3 + 6978336805440a^2c^4 + 230204396822400ab^5$$
$$- 499143392748000ab^4c + 377448582551040ab^3c^2$$
$$- 118021049546400ab^2c^3 + 13956673610880abc^4 - 423155845440ac^5$$
$$+ 38367399470400b^6 - 99828678549600b^5c + 94362145637760b^4c^2$$
$$- 39340349848800b^3c^3 + 6978336805440b^2c^4 - 423155845440bc^5$$
$$+ 4851038400c^6 + 172327780104480a^5 + 861638900522400a^4b$$
$$- 395878181418720a^4c + 1723277801044800a^3b^2$$
$$- 1583512725674880a^3bc + 316677888727200a^3c^2$$
$$+ 1723277801044800a^2b^3 - 2375269088512320a^2b^2c$$
$$+ 950033666181600a^2bc^2 - 103944009929760a^2c^3$$
$$+ 861638900522400ab^4 - 1583512725674880ab^3c$$
$$+ 950033666181600ab^2c^2 - 207888019859520abc^3$$
$$+ 12616796527200ac^4 + 172327780104480b^5 - 395878181418720b^4c$$
$$+ 316677888727200b^3c^2 - 103944009929760b^2c^3 + 12616796527200bc^4$$
$$- 370471082880c^5 + 436896549048960a^4 + 1747586196195840a^3b$$
$$- 852255645040800a^3c + 2621379294293760a^2b^2$$
$$- 2556766935122400a^2bc + 541582040039040a^2c^2$$
$$+ 1747586196195840ab^3 - 2556766935122400ab^2c$$
$$+ 1083164080078080abc^2 - 124264462284000ac^3 + 436896549048960b^4$$
$$- 852255645040800b^3c + 541582040039040b^2c^2 - 124264462284000bc^3$$
$$+ 7667057011200c^4 + 665273659050000a^3 + 1995820977150000a^2b$$
$$- 1038786858869760a^2c + 1995820977150000ab^2$$

$$
\begin{aligned}
&- 2077573717739520abc + 468198487296000ac^2 + 665273659050000b^3 \\
&- 1038786858869760b^2c + 468198487296000bc^2 - 56341814664000c^3 \\
&+ 598333974638400a^2 + 1196667949276800ab - 672080812392960ac \\
&+ 598333974638400b^2 - 672080812392960bc + 162618365840640c^2 \\
&+ 287756965782720a + 287756965782720b - 178085871187200c \\
&+ 54963381392640, \\
\mathsf{c}_3^9(a,b,c) = &-4070485118700a^6 - 24422910712200a^5b + 10860508058400a^5c \\
&- 61057276780500a^4b^2 + 54302540292000a^4bc - 10655185020480a^4c^2 \\
&- 81409702374000a^3b^3 + 108605080584000a^3b^2c \\
&- 42620740081920a^3bc^2 + 4705779078000a^3c^3 - 61057276780500a^2b^4 \\
&+ 108605080584000a^2b^3c - 63931110122880a^2b^2c^2 \\
&+ 14117337234000a^2bc^3 - 917263947600a^2c^4 - 24422910712200ab^5 \\
&+ 54302540292000ab^4c - 42620740081920ab^3c^2 \\
&+ 14117337234000ab^2c^3 - 1834527895200abc^4 + 65627452800ac^5 \\
&- 4070485118700b^6 + 10860508058400b^5c - 10655185020480b^4c^2 \\
&+ 4705779078000b^3c^3 - 917263947600b^2c^4 + 65627452800bc^5 \\
&- 1028927040c^6 - 33219988401600a^5 - 166099942008000a^4b \\
&+ 79420980038400a^4c - 332199884016000a^3b^2 \\
&+ 317683920153600a^3bc - 67068847525680a^3c^2 \\
&- 332199884016000a^2b^3 + 476525880230400a^2b^2c \\
&- 201206542577040a^2bc^2 + 23815450779120a^2c^3 \\
&- 166099942008000ab^4 + 317683920153600ab^3c \\
&- 201206542577040ab^2c^2 + 47630901558240abc^3 - 3272684688000ac^4 \\
&- 33219988401600b^5 + 79420980038400b^4c - 67068847525680b^3c^2 \\
&+ 23815450779120b^2c^3 - 3272684688000bc^4 + 119176130880c^5 \\
&- 117313721605080a^4 - 469254886420320a^3b + 241068212985600a^3c \\
&- 703882329630480a^2b^2 + 723204638956800a^2bc \\
&- 164262032494560a^2c^2 - 469254886420320ab^3
\end{aligned}
$$

$$
\begin{aligned}
&+ 723204638956800ab^2c - 328524064989120abc^2 + 41681803035600ac^3 \\
&- 117313721605080b^4 + 241068212985600b^3c - 164262032494560b^2c^2 \\
&+ 41681803035600bc^3 - 3018609628080c^4 - 224638841306880a^3 \\
&- 673916523920640a^2b + 373320214787520a^2c - 673916523920640ab^2 \\
&+ 746640429575040abc - 183208140133200ac^2 - 224638841306880b^3 \\
&+ 373320214787520b^2c - 183208140133200bc^2 + 25006507675920c^3 \\
&- 240493677884460a^2 - 480987355768920ab + 290012425466400ac \\
&- 240493677884460b^2 + 290012425466400bc - 77634243353760c^2 \\
&- 132452305876800a - 132452305876800b + 88585116215040c \\
&- 27824855803920, \\
\mathsf{c}_4^9(a,b,c) &= 2318567951700a^5 + 11592839758500a^4b - 5598140347800a^4c \\
&+ 23185679517000a^3b^2 - 22392561391200a^3bc + 4822389772200a^3c^2 \\
&+ 23185679517000a^2b^3 - 33588842086800a^2b^2c \\
&+ 14467169316600a^2bc^2 - 1777738762800a^2c^3 + 11592839758500ab^4 \\
&- 22392561391200ab^3c + 14467169316600ab^2c^2 - 3555477525600abc^3 \\
&+ 262075413600ac^4 + 2318567951700b^5 - 5598140347800b^4c \\
&+ 4822389772200b^3c^2 - 1777738762800b^2c^3 + 262075413600bc^4 \\
&- 10937908800c^5 + 14260412766600a^4 + 57041651066400a^3b \\
&- 30109542863400a^3c + 85562476599600a^2b^2 - 90328628590200a^2bc \\
&+ 21316931115480a^2c^2 + 57041651066400ab^3 - 90328628590200ab^2c \\
&+ 42633862230960abc^2 - 5734783454400ac^3 + 14260412766600b^4 \\
&- 30109542863400b^3c + 21316931115480b^2c^2 - 5734783454400bc^3 \\
&+ 458088976800c^4 + 36608314743000a^3 + 109824944229000a^2b \\
&- 63300677239800a^2c + 109824944229000ab^2 - 126601354479600abc \\
&+ 32767531556760ac^2 + 36608314743000b^3 - 63300677239800b^2c \\
&+ 32767531556760bc^2 - 4836684237840c^3 + 47452337412480a^2 \\
&+ 94904674824960ab - 60078583164600ac + 47452337412480b^2 \\
&- 60078583164600bc + 17184195985320c^2 + 29850380419860a \\
&+ 29850380419860b - 21077674385040c + 6819084974520,
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{c}_5^9(a,b,c) &= -773579631825a^4 - 3094318527300a^3b + 1620971352000a^3c \\
&\quad - 4641477790950a^2b^2 + 4862914056000a^2bc - 1148188041000a^2c^2 \\
&\quad - 3094318527300ab^3 + 4862914056000ab^2c - 2296376082000abc^2 \\
&\quad + 313718605200ac^3 - 773579631825b^4 + 1620971352000b^3c \\
&\quad - 1148188041000b^2c^2 + 313718605200bc^3 - 26207541360c^4 \\
&\quad - 3288256171200a^3 - 9864768513600a^2b + 5771815849800a^2c \\
&\quad - 9864768513600ab^2 + 11543631699600abc - 3056688835200ac^2 \\
&\quad - 3288256171200b^3 + 5771815849800b^2c - 3056688835200bc^2 \\
&\quad + 468905477040c^3 - 5476498577550a^2 - 10952997155100ab \\
&\quad + 7142749614000ac - 5476498577550b^2 + 7142749614000bc \\
&\quad - 2124667064520c^2 - 4002806808000a - 4002806808000b \\
&\quad + 2937946251240c - 994781572785, \\
\mathsf{c}_6^9(a,b,c) &= 156307951800a^3 + 468923855400a^2b - 266302436400a^2c \\
&\quad + 468923855400ab^2 - 532604872800abc + 137782564920ac^2 \\
&\quad + 156307951800b^3 - 266302436400b^2c + 137782564920bc^2 \\
&\quad - 20914573680c^3 + 402348246300a^2 + 804696492600ab \\
&\quad - 526815689400ac + 402348246300b^2 - 526815689400bc \\
&\quad + 157851734040c^2 + 357771534120a + 357771534120b \\
&\quad - 268930541880c + 97616659140, \\
\mathsf{c}_7^9(a,b,c) &= -18814846050a^2 - 37629692100ab + 23156733600ac \\
&\quad - 18814846050b^2 + 23156733600bc - 6561074520c^2 - 23156733600a \\
&\quad - 23156733600b + 17367550200c - 7043506470, \\
\mathsf{c}_8^9(a,b,c) &= -827026200c + 413513100 + 1240539300a + 1240539300b, \\
\mathsf{c}_9^9(a,b,c) &= -34459425.
\end{aligned}
$$

C.0.4 Hypergeometric Series Cases $m = 10$

For $m = 10$ we can write

$$\mathscr{R}_{10}(X; a, b, c) = \sum_{l=1}^{10} \mathsf{c}_l^{10} X^l = \sum_{l=1}^{10} \frac{\mathsf{q}_l^{10}(a, b, c)}{2^{10}\mathscr{A}_{10}(c)} X^l \tag{C.8}$$

where $\mathscr{A}_{10}(c) = c(c+1)(c+2)\cdots(c+9)$ and

$$\begin{aligned}
\mathsf{c}_1^{10}(a, b, c) = {} & -237588086736000a^9 - 2138292780624000a^8b \\
& + 791960289120000a^8c - 8553171122496000a^7b^2 \\
& + 6335682312960000a^7bc - 1029548375856000a^7c^2 \\
& - 19957399285824000a^6b^3 + 22174888095360000a^6b^2c \\
& - 7206838630992000a^6bc^2 + 658709827776000a^6c^3 \\
& - 29936098928736000a^5b^4 + 44349776190720000a^5b^3c \\
& - 21620515892976000a^5b^2c^2 + 3952258966656000a^5bc^3 \\
& - 215295871190400a^5c^4 - 29936098928736000a^4b^5 \\
& + 55437220238400000a^4b^4c - 36034193154960000a^4b^3c^2 \\
& + 9880647416640000a^4b^2c^3 - 1076479355952000a^4bc^4 \\
& + 33811637932800a^4c^5 - 19957399285824000a^3b^6 \\
& + 44349776190720000a^3b^5c - 36034193154960000a^3b^4c^2 \\
& + 13174196555520000a^3b^3c^3 - 2152958711904000a^3b^2c^4 \\
& + 135246551731200a^3bc^5 - 2143293425280a^3c^6 \\
& - 8553171122496000a^2b^7 + 22174888095360000a^2b^6c \\
& - 21620515892976000a^2b^5c^2 + 9880647416640000a^2b^4c^3 \\
& - 2152958711904000a^2b^3c^4 + 202869827596800a^2b^2c^5 \\
& - 6429880275840a^2bc^6 + 37125258240a^2c^7 - 2138292780624000ab^8 \\
& + 6335682312960000ab^7c - 7206838630992000ab^6c^2 \\
& + 3952258966656000ab^5c^3 - 1076479355952000ab^4c^4 \\
& + 135246551731200ab^3c^5 - 6429880275840ab^2c^6 + 74250516480abc^7 \\
& - 67108608ac^8 - 237588086736000b^9 + 791960289120000b^8c \\
& - 1029548375856000b^7c^2 + 658709827776000b^6c^3
\end{aligned}$$

$$- 2152958711904000b^5c^4 + 338116379328000b^4c^5 - 2143293425280b^3c^6$$
$$+ 37125258240b^2c^7 - 67108608bc^8 + 512c^9 - 3563821301040000a^8$$
$$- 28510570408320000a^7b + 11008248018768000a^7c$$
$$- 99786996429120000a^6b^2 + 77057736131376000a^6bc$$
$$- 13018318657344000a^6c^2 - 199573992858240000a^5b^3$$
$$+ 231173208394128000a^5b^2c - 78109911944064000a^5bc^2$$
$$+ 7374030247584000a^5c^3 - 249467491072800000a^4b^4$$
$$+ 385288680656880000a^4b^3c - 195274779860160000a^4b^2c^2$$
$$+ 36870151237920000a^4bc^3 - 2046030740208000a^4c^4$$
$$- 199573992858240000a^3b^5 + 385288680656880000a^3b^4c$$
$$- 260366373146880000a^3b^3c^2 + 73740302475840000a^3b^2c^3$$
$$- 8184122960832000a^3bc^4 + 254527935742080a^3c^5$$
$$- 99786996429120000a^2b^6 + 231173208394128000a^2b^5c$$
$$- 195274779860160000a^2b^4c^2 + 73740302475840000a^2b^3c^3$$
$$- 12276184441248000a^2b^2c^4 + 763583807226240a^2bc^5$$
$$- 11300499705600a^2c^6 - 28510570408320000ab^7$$
$$+ 77057736131376000ab^6c - 78109911944064000ab^5c^2$$
$$+ 36870151237920000ab^4c^3 - 8184122960832000ab^3c^4$$
$$+ 763583807226240ab^2c^5 - 22600999411200abc^6 + 108422995968ac^7$$
$$- 3563821301040000b^8 + 11008248018768000b^7c$$
$$- 13018318657344000b^6c^2 + 7374030247584000b^5c^3$$
$$- 2046030740208000b^4c^4 + 254527935742080b^3c^5$$
$$- 11300499705600b^2c^6 + 108422995968bc^7 - 67085568c^8$$
$$- 24233984847072000a^7 - 169637893929504000a^6b$$
$$+ 68158868057280000a^6c - 508913681788512000a^5b^2$$
$$+ 408953208343680000a^5bc - 71648940675312000a^5c^2$$
$$- 848189469647520000a^4b^3 + 1022383020859200000a^4b^2c$$
$$- 358244703376560000a^4bc^2 + 34789722222528000a^4c^3$$
$$- 848189469647520000a^3b^4 + 1363177361145600000a^3b^3c$$
$$- 716489406753120000a^3b^2c^2 + 139158888890112000a^3bc^3$$

$$- 7805634165504000a^3c^4 - 508913681788512000a^2b^5$$
$$+ 1022383020859200000a^2b^4c - 716489406753120000a^2b^3c^2$$
$$+ 208738333335168000a^2b^2c^3 - 23416902496512000a^2bc^4$$
$$+ 709423607746560a^2c^5 - 169637893929504000ab^6$$
$$+ 408953208343680000ab^5c - 358244703376560000ab^4c^2$$
$$+ 139158888890112000ab^3c^3 - 23416902496512000ab^2c^4$$
$$+ 1418847215493120abc^5 - 18953876850048ac^6$$
$$- 24233984847072000b^7 + 68158868057280000b^6c$$
$$- 71648940675312000b^5c^2 + 34789722222528000b^4c^3$$
$$- 7805634165504000b^3c^4 + 709423607746560b^2c^5$$
$$- 18953876850048bc^6 + 71298183168c^7 - 97094331446112000a^6$$
$$- 582565988676672000a^5b + 243430993757952000a^5c$$
$$- 1456414971691680000a^4b^2 + 1217154968789760000a^4bc$$
$$- 220771596623760000a^4c^2 - 1941886628922240000a^3b^3$$
$$+ 2434309937579520000a^3b^2c - 883086386495040000a^3bc^2$$
$$+ 87849255969820800a^3c^3 - 1456414971691680000a^2b^4$$
$$+ 2434309937579520000a^2b^3c - 1324629579742560000a^2b^2c^2$$
$$+ 263547767909462400a^2bc^3 - 14794590551126400a^2c^4$$
$$- 582565988676672000ab^5 + 1217154968789760000ab^4c$$
$$- 883086386495040000ab^3c^2 + 263547767909462400ab^2c^3$$
$$- 29589181102252800abc^4 + 853171097401728ac^5$$
$$- 97094331446112000b^6 + 243430993757952000b^5c$$
$$- 220771596623760000b^4c^2 + 87849255969820800b^3c^3$$
$$- 14794590551126400b^2c^4 + 853171097401728bc^5 - 9796665731328c^6$$
$$- 250243612156137600a^5 - 1251218060780688000a^4b$$
$$+ 544160234437747200a^4c - 2502436121561376000a^3b^2$$
$$+ 2176640937750988800a^3bc - 408353857714494720a^3c^2$$
$$- 2502436121561376000a^2b^3 + 3264961406626483200a^2b^2c$$
$$- 1225061573143484160a^2bc^2 + 124242253927599360a^2c^3$$
$$- 1251218060780688000ab^4 + 2176640937750988800ab^3c$$

$$
\begin{aligned}
&- 1225061573143484160ab^2c^2 + 248484507855198720abc^3\\
&- 13764289009387392ac^4 - 250243612156137600b^5\\
&+ 544160234437747200b^4c - 408353857714494720b^3c^2\\
&+ 124242253927599360b^2c^3 - 13764289009387392bc^4\\
&+ 364463819860224c^5 - 426211428687408000a^4\\
&- 1704845714749632000a^3b + 773632579799963520a^3c\\
&- 2557268572124448000a^2b^2 + 2320897739399890560a^2bc\\
&- 450295223849251200a^2c^2 - 1704845714749632000ab^3\\
&+ 2320897739399890560ab^2c - 900590447698502400abc^2\\
&+ 92517178199225472ac^3 - 426211428687408000b^4\\
&+ 773632579799963520b^3c - 450295223849251200b^2c^2\\
&+ 92517178199225472bc^3 - 4944597616852992c^4\\
&- 474863475305318400a^3 - 1424590425915955200a^2b\\
&+ 677999421641283840a^2c - 1424590425915955200ab^2\\
&+ 1355998843282567680abc - 272301046196884992ac^2\\
&- 474863475305318400b^3 + 677999421641283840b^2c\\
&- 272301046196884992bc^2 + 28049778809251072c^3\\
&- 329857232436748800a^2 - 659714464873497600ab\\
&+ 332479439629811712ac - 329857232436748800b^2\\
&+ 332479439629811712bc - 69248253231770112c^2\\
&- 127764900985559040a - 127764900985559040b\\
&+ 69440777306671104c - 20616747323719680,
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{c}_2^{10}(a,b,c) &= 672129154897200a^8 + 5377033239177600a^7b\\
&- 2152434928644000a^7c + 18819616337121600a^6b^2\\
&- 15067044500508000a^6bc + 2669471860255200a^6c^2\\
&+ 37639232674243200a^5b^3 - 45201133501524000a^5b^2c\\
&+ 16016831161531200a^5bc^2 - 1613839078051200a^5c^3\\
&+ 47049040842804000a^4b^4 - 75335222502540000a^4b^3c\\
&+ 40042077903828000a^4b^2c^2 - 8069195390256000a^4bc^3\\
&+ 491592239218080a^4c^4 + 37639232674243200a^3b^5
\end{aligned}
$$

$$- 75335222502540000a^3b^4c + 53389437205104000a^3b^3c^2$$
$$- 16138390780512000a^3b^2c^3 + 1966368956872320a^3bc^4$$
$$- 70440912360000a^3c^5 + 18819616337121600a^2b^6$$
$$- 45201133501524000a^2b^5c + 40042077903828000a^2b^4c^2$$
$$- 16138390780512000a^2b^3c^3 + 2949553435308480a^2b^2c^4$$
$$- 211322737080000a^2bc^5 + 3929371279680a^2c^6 + 5377033239177600ab^7$$
$$- 15067044500508000ab^6c + 16016831161531200ab^5c^2$$
$$- 8069195390256000ab^4c^3 + 1966368956872320ab^3c^4$$
$$- 211322737080000ab^2c^5 + 7858742559360abc^6 - 55687887360ac^7$$
$$+ 672129154897200b^8 - 2152434928644000b^7c + 2669471860255200b^6c^2$$
$$- 1613839078051200b^5c^3 + 491592239218080b^4c^4 - 70440912360000b^3c^5$$
$$+ 3929371279680b^2c^6 - 55687887360bc^7 + 67108608c^8$$
$$+ 8735604831954000a^7 + 61149233823678000a^6b$$
$$- 25770953493885600a^6c + 183447701471034000a^5b^2$$
$$- 154625720963313600a^5bc + 28806235582924800a^5c^2$$
$$+ 305746169118390000a^4b^3 - 386564302408284000a^4b^2c$$
$$+ 144031177914624000a^4bc^2 - 15190594495876800a^4c^3$$
$$+ 305746169118390000a^3b^4 - 515419069877712000a^3b^3c$$
$$+ 288062355829248000a^3b^2c^2 - 60762377983507200a^3bc^3$$
$$+ 3831972299719200a^3c^4 + 183447701471034000a^2b^5$$
$$- 386564302408284000a^2b^4c + 288062355829248000a^2b^3c^2$$
$$- 91143566975260800a^2b^2c^3 + 11495916899157600a^2bc^4$$
$$- 415917091961280a^2c^5 + 61149233823678000ab^6$$
$$- 154625720963313600ab^5c + 144031177914624000ab^4c^2$$
$$- 60762377983507200ab^3c^3 + 11495916899157600ab^2c^4$$
$$- 831834183922560abc^5 + 14807456133120ac^6 + 8735604831954000b^7$$
$$- 25770953493885600b^6c + 28806235582924800b^5c^2$$
$$- 15190594495876800b^4c^3 + 3831972299719200b^3c^4$$
$$- 415917091961280b^2c^5 + 14807456133120bc^6 - 89860366848c^7$$
$$+ 51054851409962400a^6 + 306329108459774400a^5b$$

$$- 1357463933334552000a^5c + 765822771149436000a^4b^2$$
$$- 6787319666727760000a^4bc + 1327685318583624000a^4c^2$$
$$+ 1021097028199248000a^3b^3 - 13574639333345520000a^3b^2c$$
$$+ 5310741274334496000a^3bc^2 - 584990222675040000a^3c^3$$
$$+ 765822771149436000a^2b^4 - 13574639333345520000a^2b^3c$$
$$+ 7966111911501744000a^2b^2c^2 - 1754970668025120000a^2bc^3$$
$$+ 114027550965048000a^2c^4 + 306329108459774400ab^5$$
$$- 6787319666727760000ab^4c + 5310741274334496000ab^3c^2$$
$$- 1754970668025120000ab^2c^3 + 228055101930096000abc^4$$
$$- 8236956583497600ac^5 + 51054851409962400b^6$$
$$- 1357463933334552000b^5c + 1327685318583624000b^4c^2$$
$$- 584990222675040000b^3c^3 + 114027550965048000b^2c^4$$
$$- 8236956583497600bc^5 + 136608425681280c^6 + 173197755429098400a^5$$
$$+ 865988777145492000a^4b - 4037840123020704000a^4c$$
$$+ 1731977554290984000a^3b^2 - 16151360492082816000a^3bc$$
$$+ 3318309833824440000a^3c^2 + 1731977554290984000a^2b^3$$
$$- 24227040738124224000a^2b^2c + 9954929501473320000a^2bc^2$$
$$- 1144029086299008000a^2c^3 + 865988777145492000ab^4$$
$$- 16151360492082816000ab^3c + 9954929501473320000ab^2c^2$$
$$- 2288058172598016000abc^3 + 152387644696056000ac^4$$
$$+ 173197755429098400b^5 - 4037840123020704000b^4c$$
$$+ 3318309833824440000b^3c^2 - 1144029086299008000b^2c^3$$
$$+ 152387644696056000bc^4 - 5408806161521280c^5$$
$$+ 3687962396293375200a^4 + 14751849585173500800a^3b$$
$$- 72586846979665200000a^3c + 22127774377760251200a^2b^2$$
$$- 217760540938995600000a^2bc + 47093634434092512000a^2c^2$$
$$+ 14751849585173500800ab^3 - 217760540938995600000ab^2c$$
$$+ 94187268868185024000abc^2 - 11293299303439104000ac^3$$
$$+ 3687962396293375200b^4 - 72586846979665200000b^3c$$
$$+ 47093634434092512000b^2c^2 - 11293299303439104000bc^3$$

$$
\begin{aligned}
&+ 7660756232368512c^4 + 4986807748921908000a^3 \\
&+ 14960423246765724000a^2b - 7815402863564587200a^2c \\
&+ 14960423246765724000ab^2 - 15630805727129174400abc \\
&+ 3575583394759252800ac^2 + 4986807748921908000b^3 \\
&- 7815402863564587200b^2c + 3575583394759252800bc^2 \\
&- 447917928888097920c^3 + 4121605616581152000a^2 \\
&+ 8243211233162304000ab - 4622329686257942400ac \\
&+ 4121605616581152000b^2 - 4622329686257942400bc \\
&+ 1128486255920661120c^2 + 1865646346307184000a \\
&+ 1865646346307184000b - 1145662267938693120c \\
&+ 342078236158579200,
\end{aligned}
$$

$$
\begin{aligned}
\mathsf{c}_3^{10}(a,b,c) &= -7678007862525000a^7 - 53746055037675000a^6b \\
&+ 23201765176590000a^6c - 161238165113025000a^5b^2 \\
&+ 139210591059540000a^5bc - 26825454904248000a^5c^2 \\
&- 268730275188375000a^4b^3 + 348026477648850000a^4b^2c \\
&- 134127274521240000a^4bc^2 + 14857566115392000a^4c^3 \\
&- 268730275188375000a^3b^4 + 464035303531800000a^3b^3c \\
&- 268254549042480000a^3b^2c^2 + 59430264461568000a^3bc^3 \\
&- 4036797584820000a^3c^4 - 161238165113025000a^2b^5 \\
&+ 348026477648850000a^2b^4c - 268254549042480000a^2b^3c^2 \\
&+ 89145396692352000a^2b^2c^3 - 12110392754460000a^2bc^4 \\
&+ 493086386520000a^2c^5 - 53746055037675000ab^6 \\
&+ 139210591059540000ab^5c - 134127274521240000ab^4c^2 \\
&+ 59430264461568000ab^3c^3 - 12110392754460000ab^2c^4 \\
&+ 986172773040000abc^5 - 21432934252800ac^6 - 7678007862525000b^7 \\
&+ 23201765176590000b^6c - 26825454904248000b^5c^2 \\
&+ 14857566115392000b^4c^3 - 4036797584820000b^3c^4 + 493086386520000b^2c^5 \\
&- 21432934252800bc^6 + 185626291200c^7 - 82924134832539000a^6 \\
&- 497544808995234000a^5b + 228560367979944000a^5c \\
&- 1243862022488085000a^4b^2 + 1142801839899720000a^4bc
\end{aligned}
$$

$$
\begin{aligned}
&- 234362838934224000a^4c^2 - 165848269665078000a^3b^3\\
&+ 228560367979944000a^3b^2c - 93745135573689600a^3bc^2\\
&+ 11021300697607200a^3c^3 - 124386202248808500a^2b^4\\
&+ 228560367979944000a^2b^3c - 140617703360534400a^2b^2c^2\\
&+ 33063902092821600a^2bc^3 - 2363025067537680a^2c^4\\
&- 49754480899523400ab^5 + 114280183989972000ab^4c\\
&- 93745135573689600ab^3c^2 + 33063902092821600ab^2c^3\\
&- 4726050135075360abc^4 + 198175205854080ac^5 - 8292413483253900b^6\\
&+ 22856036797994400b^5c - 23436283893422400b^4c^2\\
&+ 11021300697607200b^3c^3 - 2363025067537680b^2c^4\\
&+ 198175205854080bc^5 - 4221387569280c^6 - 39690362682570600a^5\\
&- 198451813412853000a^4b + 96903628944922800a^4c\\
&- 396903626825706000a^3b^2 + 387614515779691200a^3bc\\
&- 84539340984898800a^3c^2 - 396903626825706000a^2b^3\\
&+ 581421773669536800a^2b^2c - 253618022954696400a^2bc^2\\
&+ 31623180048532800a^2c^3 - 198451813412853000ab^4\\
&+ 387614515779691200ab^3c - 253618022954696400ab^2c^2\\
&+ 63246360097065600abc^3 - 4745438722508400ac^4\\
&- 39690362682570600b^5 + 96903628944922800b^4c\\
&- 84539340984898800b^3c^2 + 31623180048532800b^2c^3\\
&- 4745438722508400bc^4 + 203341150592160c^5 - 107444293901544600a^4\\
&- 429777175606178400a^3b + 223459661837412000a^3c\\
&- 644665763409267600a^2b^2 + 670378985512236000a^2bc\\
&- 155802048328508400a^2c^2 - 429777175606178400ab^3\\
&+ 670378985512236000ab^2c - 311604096657016800abc^2\\
&+ 41272813651960800ac^3 - 107444293901544600b^4\\
&+ 223459661837412000b^3c - 155802048328508400b^2c^2\\
&+ 41272813651960800bc^3 - 3247981848375600c^4\\
&- 174884795836650900a^3 - 524654387509952700a^2b\\
&+ 291954144335981400a^2c - 524654387509952700ab^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 583908288671962800abc - 145359620462311920ac^2\\
&- 174884795836650900b^3 + 291954144335981400b^2c\\
&- 145359620462311920bc^2 + 20543073379241280c^3\\
&- 168016722219129420a^2 - 336033444438258840ab\\
&+ 202135396105581120ac - 168016722219129420b^2\\
&+ 202135396105581120bc - 54433907314363920c^2\\
&- 85939962622340400a - 85939962622340400b\\
&+ 56930339398141440c - 17227866774157200,\\
\mathsf{c}_4^{10}(a,b,c) &= 473814336996000a^6 + 2842886021976000a^5b\\
&- 1321583732469000a^5c + 7107215054940000a^4b^2\\
&- 6607918662345000a^4bc + 1382740665906600a^4c^2\\
&+ 9476286739920000a^3b^3 - 13215837324690000a^3b^2c\\
&+ 5530962663626400a^3bc^2 - 672432949188000a^3c^3\\
&+ 7107215054940000a^2b^4 - 13215837324690000a^2b^3c\\
&+ 8296443995439600a^2b^2c^2 - 2017298847564000a^2bc^3\\
&+ 152501242093200a^2c^4 + 2842886021976000ab^5\\
&- 6607918662345000ab^4c + 5530962663626400ab^3c^2\\
&- 2017298847564000ab^2c^3 + 305002484186400abc^4\\
&- 14088182472000ac^5 + 473814336996000b^6 - 1321583732469000b^5c\\
&+ 1382740665906600b^4c^2 - 672432949188000b^3c^3\\
&+ 152501242093200b^2c^4 - 14088182472000bc^5 + 357215570880c^6\\
&+ 4052786068831500a^5 + 20263930344157500a^4b\\
&- 10149363942517800a^4c + 40527860688315000a^3b^2\\
&- 40597455770071200a^3bc + 9165620412748800a^3c^2\\
&+ 40527860688315000a^2b^3 - 60896183655106800a^2b^2c\\
&+ 27496861238246400a^2bc^2 - 3603028223995200a^2c^3\\
&+ 20263930344157500ab^4 - 40597455770071200ab^3c\\
&+ 27496861238246400ab^2c^2 - 7206056447990400abc^3\\
&+ 583392969432000ac^4 + 4052786068831500b^5\\
&- 10149363942517800b^4c + 9165620412748800b^3c^2
\end{aligned}
$$

$$
\begin{aligned}
& - 3603028223995200b^2c^3 + 583392969432000bc^4 - 28333140151680c^5 \\
& + 15008574575194200a^4 + 60034298300776800a^3b \\
& - 32353536759944400a^3c + 90051447451165200a^2b^2 \\
& - 97060610279833200a^2bc + 23636990654877600a^2c^2 \\
& + 60034298300776800ab^3 - 97060610279833200ab^2c \\
& + 47273981309755200abc^2 - 6680682791546400ac^3 \\
& + 15008574575194200b^4 - 32353536759944400b^3c \\
& + 23636990654877600b^2c^2 - 6680682791546400bc^3 \\
& + 579397874138160c^4 + 30125085166776600a^3 \\
& + 90375255500329800a^2b - 52554320667248400a^2c \\
& + 90375255500329800ab^2 - 105108641334496800abc \\
& + 27714281447361600ac^2 + 30125085166776600b^3 \\
& - 52554320667248400b^2c + 27714281447361600bc^2 \\
& - 4242871696190400c^3 + 33739331082256800a^2 \\
& + 67478662164513600ab - 42678376665561000ac \\
& + 33739331082256800b^2 - 42678376665561000bc \\
& + 12291699740059320c^2 + 19345684958718300a \\
& + 19345684958718300b - 13525451207278920c + 4184827092651240, \\
\mathsf{c}_5^{10}(a,b,c) &= -176334908124375a^5 - 881674540621875a^4b \\
& + 440940390140250a^4c - 1763349081243750a^3b^2 \\
& + 1763761560561000a^3bc - 400379923944000a^3c^2 \\
& - 1763349081243750a^2b^3 + 2645642340841500a^2b^2c \\
& - 1201139771832000a^2bc^2 + 160103083140000a^2c^3 \\
& - 881674540621875ab^4 + 1763761560561000ab^3c \\
& - 1201139771832000ab^2c^2 + 320206166280000abc^3 \\
& - 26911983898800ac^4 - 176334908124375b^5 + 440940390140250b^4c \\
& - 400379923944000b^3c^2 + 160103083140000b^2c^3 - 26911983898800bc^4 \\
& + 1408818247200c^5 - 1123721332358625a^4 - 4494885329434500a^3b \\
& + 2459476675656000a^3c - 6742327994151750a^2b^2 \\
& + 7378430026968000a^2bc - 1837201211445600a^2c^2
\end{aligned}
$$

$$
\begin{aligned}
&- 4494885329434500ab^3 + 7378430026968000ab^2c \\
&- 3674402422891200abc^2 + 537506381412000ac^3 \\
&- 1123721332358625b^4 + 2459476675656000b^3c \\
&- 1837201211445600b^2c^2 + 537506381412000bc^3 - 49368635643600c^4 \\
&- 2986291331273250a^3 - 8958873993819750a^2b \\
&+ 5352194564016300a^2c - 8958873993819750ab^2 \\
&+ 10704389128032600abc - 2923580071011600ac^2 \\
&- 2986291331273250b^3 + 5352194564016300b^2c \\
&- 2923580071011600bc^2 + 470327527706400c^3 - 3998044170870750a^2 \\
&- 7996088341741500ab + 5238398285920800ac - 3998044170870750b^2 \\
&+ 5238398285920800bc - 1578978286376400c^2 - 2585968368758775a \\
&- 2585968368758775b + 1882016702379450c - 602013206401425, \\
\mathsf{c}_6^{10}(a,b,c) &= 41426672762475a^4 + 165706691049900a^3b - 89095532526000a^3c \\
&+ 248560036574850a^2b^2 - 267286597578000a^2bc \\
&+ 65776701790800a^2c^2 + 165706691049900ab^3 \\
&- 267286597578000ab^2c + 131553403581600abc^2 \\
&- 19212369976800ac^3 + 41426672762475b^4 - 89095532526000b^3c \\
&+ 65776701790800b^2c^2 - 19212369976800bc^3 + 1794132259920c^4 \\
&+ 180665940955500a^3 + 541997822866500a^2b - 325473679931400a^2c \\
&+ 541997822866500ab^2 - 650947359862800abc + 179510998867200ac^2 \\
&+ 180665940955500b^3 - 325473679931400b^2c + 179510998867200bc^2 \\
&- 29429635435200c^3 + 308074909492350a^2 + 616149818984700ab \\
&- 411746020686000ac + 308074909492350b^2 - 411746020686000bc \\
&+ 127251397072800c^2 + 229644913598100a + 229644913598100b \\
&- 172064546854200c + 57785995322055, \\
\mathsf{c}_7^{10}(a,b,c) &= -6187189758750a^3 - 18561569276250a^2b + 10724462248500a^2c \\
&- 18561569276250ab^2 + 21448924497000abc - 5719713199200ac^2 \\
&- 6187189758750b^3 + 10724462248500b^2c - 5719713199200bc^2 \\
&+ 914874760800c^3 - 16196687857350a^2 - 32393375714700ab \\
&+ 21558918981600ac - 16196687857350b^2 + 21558918981600bc
\end{aligned}
$$

$$
\begin{aligned}
&- 6641571736800c^2 - 14601767830650a - 14601767830650b \\
&+ 11124283470300c - 4016108146050, \\
\mathsf{c}_8^{10}(a,b,c) &= 569614295250a^2 + 1139228590500ab - 707107401000ac \\
&+ 569614295250b^2 - 707107401000bc + 204275471400c^2 \\
&+ 707107401000a + 707107401000b - 534258925200c \\
&+ 216060594750, \\
\mathsf{c}_9^{10}(a,b,c) &= 19641872250c - 9820936125 - 29462808375a - 29462808375b, \\
\mathsf{c}_{10}^{10}(a,b,c) &= 654729075.
\end{aligned}
$$

Appendix D

Conclusion

This concludes the research project. In the second chapter, derivations of the trace and determinant of triangular matrix systems of linear combinations of hypergeometric series were established. The connection to more generalised hypergeometric series as the dimensions of the matrix blow-up to infinity was then demonstrated. The chapter was finalized by deriving the determinant and trace on five rank-one symmetric spaces, namely the $n-$dimensional sphere, and the real, complex, hyperbolic and Cayley projective spaces. Working on rank-one symmetric spaces restricts us to working on $\mathbb{S}^n$ and $\mathbb{K}\mathbf{P}^n$ for a vector space $\mathbb{K}$. No such results on the trace and determinant or infinite blow-up of the limits exist on symmetric spaces of rank two or above. This presents a possible future research opportunity.

The third chapter was built upon the work in the second chapter by obtaining a similar inner product matrix system for the space of Jacobi polynomials as well as the generalised determinant and trace. Examples of the hypergeometric series in two variables (namely X_1 and X_2) are obtained as well as the trace and determinant. By writing these polynomials as a linear combination polynomials in one variable each, a derivation of matrix systems was again possible demonstrating a close link with more generalised hypergeometric series. At the end of the chapter, the more general dummy function f was replaced with more interesting functions including infinite Fourier series over $\mathbb{Z}$. We are left with the potential prospect of examining the results on infinite Fourier series on different spaces, including $n-$dimensional tori and $n-$dimensional square/circular summation.

In the fourth chapter, a transition was initiated by moving onto a closely related but separate topic, namely spherical twists for a variational problem with energy functional

$\mathbb{F}[u, \Omega]$. Problems on multiplicity and solving the Euler-Lagrange equations were addressed and dealt with. By choosing $u(x)$ as variations of the twist $\mathbf{Q}\theta$ including $f(r)\mathbf{Q}(\mathbf{r})\mathbf{v}(\mathbf{x})$ for the restricted energy, $\mathbf{Q}(\mathbf{r})\mathbf{v}(\mathbf{x})$ in the general case, $u = \mathbf{Q}\theta$ (where $\mathbf{Q}$ is a twice continuously differentiable twist path), the exponential twist case $u = \exp(\mathscr{G}\mathbf{H})\mathbf{v}(\mathbf{x})$ and $u = \exp(\mathscr{G}(r)\mathbf{H}\theta)$, derivations of the differential equation systems that allow the Euler-Lagrange equation to be solved in both the constrained and unconstrained cases became possible. Specific cases of u were examined in this chapter in the attempt to solve the Euler-Lagrange equation. Identities parallel to the examples provided in this chapter can be established by introducing further products of twists, exponential functions, and other mathematical objects yet to be confirmed.

The work in the fifth chapter was built upon the work in the fourth chapter by extending the key calculus-based identities of our arbitrary functions A and B to spherical whirls, an $n-$dimensional extension of the spherical twist for $u = \mathbf{Q}\theta$. This was achieved in both the constrained and unconstrained cases. Given the specific choice of u in this chapter, further alternate selections of u from the previous chapter may be selected to establish higher-dimensional spherical whirl generalisations.

In the appendix, polynomials were calculated for Gengenbauer polynomials, Jacobi polynomials and hypergeometric series up to the case $m = 10$ using computational methods, and substitutions for the variables α, β, a, b, c. This brings the thesis to a close. Cases of these orthogonal polynomials above $m = 10$ do not exist presenting an opportunity for further cases $m > 10$ to be established.

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